Indirect approach to discrete-time optimal control CHEATSHEET

LQ-optimal control, algebraic Riccati equation, Hamiltonian equations

1 General cost and general nonlinear and time-varying system

$$\underset{\mathbf{x}_{i+1},\ldots,\mathbf{x}_N,\mathbf{u}_i,\ldots,\mathbf{u}_{N-1}}{\text{minimize}} \left[\phi(\mathbf{x}_N, N) + \sum_{k=i}^{N-1} L_k(\mathbf{x}_k, \mathbf{u}_k)\right]$$
$$\text{subject to } \mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k), \quad \mathbf{x}_i \text{ is given}$$

No bounds on the controls or states considered here.

1.1 Hamiltonian

Auxiliary (and very useful) function (sign change with respect to the conventions in physics)

$$H_k(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\lambda}_{k+1}) = L_k(\mathbf{x}_k, \mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^{\mathrm{T}} \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)$$

where the Lagrange variables $\boldsymbol{\lambda}_k$s are called *co-state variables*.

1.2 First-order necessary conditions

$$\begin{aligned}
\mathbf{x}_{k+1} &= \nabla_{\boldsymbol{\lambda}_{k+1}} H_k, \quad k = i, \ldots, N-1,\\
\boldsymbol{\lambda}_k &= \nabla_{\mathbf{x}_k} H_k, \quad k = i+1, \ldots, N-1\\
0 &= \nabla_{\mathbf{u}_k} H_k, \quad k = i, \ldots, N-1\\
0 &= (\nabla_{\mathbf{x}_N} \phi - \lambda_N)^{\mathrm{T}}\, \mathrm{d}\mathbf{x}_N,\\
0 &= (\nabla_{\mathbf{x}_i} H_i)^{\mathrm{T}}\, \mathrm{d}\mathbf{x}_i
\end{aligned}$$

where the first two sets of equations are discrete-time (or recurrent) equations. The third set of equations is called *stationarity* equations. The last two (blue) equations are boundary equations (at final and initial time).

Expanding the Hamiltonian, the necessary conditions are

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k), \quad k = i, \ldots, N-1,\\
\boldsymbol{\lambda}_k &= \nabla_{\mathbf{x}_k} \mathbf{f}_k\, \boldsymbol{\lambda}_{\mathbf{k+1}} + \nabla_{\mathbf{x}_k} L_k, \quad k = i+1, \ldots, N-1\\
0 &= \nabla_{\mathbf{u}_k} \mathbf{f}_k\, \boldsymbol{\lambda}_{k+1} + \nabla_{u_k} L_k, \quad k = i, \ldots, N-1\\
0 &= (\nabla_{\mathbf{x}_N} \phi - \lambda_N)^{\mathrm{T}}\, \mathrm{d}\mathbf{x}_N,\\
0 &= (\nabla_{\mathbf{x}_i} H_i)^{\mathrm{T}}\, \mathrm{d}\mathbf{x}_i
\end{aligned}$$

2 LQ-optimal regulation over a finite time interval

Linear system (below even time-invariant, but could be time-varying), quadratic cost.

$$\begin{aligned}
\underset{\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{u}_0,\ldots,\mathbf{u}_{N-1}}{\text{minimize}} \quad & \frac{1}{2}\mathbf{x}_N^{\mathrm{T}}\mathbf{S}_N\mathbf{x}_N + \frac{1}{2}\sum_{k=0}^{N-1}\left[\mathbf{x}_k^{\mathrm{T}}\mathbf{Q}\mathbf{x}_k + \mathbf{u}_k^{\mathrm{T}}\mathbf{R}\mathbf{u}_k\right]\\
\text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \qquad \mathbf{x}_0 \text{ given},\\
& \mathbf{S}_N \geq 0, \mathbf{Q} \geq 0, \mathbf{R} > 0.
\end{aligned}$$

Hamiltonian for LQ-optimal regulation

$$H_k = \frac{1}{2}\left(\mathbf{x}_k^{\mathrm{T}}\mathbf{Q}\mathbf{x}_k + \mathbf{u}_k^{\mathrm{T}}\mathbf{R}\mathbf{u}_k\right) + \boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\left(\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k\right)$$

First-order necessary conditions (state, co-state and stationarity equations, and boundary conditions, assuming that the initial state is fixed)

$$\begin{aligned}
\mathbf{x}_{k+1} &= \nabla_{\boldsymbol{\lambda}_{k+1}} H_k = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k,\\
\boldsymbol{\lambda}_k &= \nabla_{\mathbf{x}_k} H_k = \mathbf{Q}\mathbf{x}_k + \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1},\\
\mathbf{0} &= \nabla_{\mathbf{u}_k} H_k = \mathbf{R}\mathbf{u}_k + \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1},\\
0 &= (\mathbf{S}_N\mathbf{x}_N - \boldsymbol{\lambda}_N)^{\mathrm{T}}\, \mathrm{d}\mathbf{x}_N,\\
\mathbf{x}_0 &= \mathbf{r}_0.
\end{aligned}$$

If $\mathbf{R} > 0$, the third equation (the stationarity equation) is

$$\mathbf{u}_k = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}$$

2.1 Fixed final-state

$$\mathbf{x}_N = \mathbf{r}_N$$

Replaces the final-time boundary condition.

$$\begin{aligned}
\mathbf{x}_{k+1} &= \nabla_{\boldsymbol{\lambda}_{k+1}} H_k = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k,\\
\boldsymbol{\lambda}_k &= \nabla_{\mathbf{x}_k} H_k = \mathbf{Q}\mathbf{x}_k + \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1},\\
\mathbf{0} &= \nabla_{\mathbf{u}_k} H_k = \mathbf{R}\mathbf{u}_k + \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1},\\
\mathbf{x}_N &= \mathbf{r}_N,\\
\mathbf{x}_0 &= \mathbf{r}_0.
\end{aligned}$$

Numerical solution possible (shooting).

Analytical solution possible for $\mathbf{Q} = \mathbf{0}$ (*the minimum-energy problem*).

$$\mathbf{u}_k = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})^{N-k-1}\mathbf{G}_{0,N,R}^{-1}\left(\mathbf{r}_N - \mathbf{A}^N\mathbf{x}_0\right)$$

where $\mathbf{G}_{0,N,R}$ is *weighted reachability Gramian.*

$$G_{0,N,R} = \sum_{i=0}^{N-1} \mathbf{A}^{N-1-i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})^{N-i-1}$$

Open-loop (precomputed) control sequence. Proportional to the difference $(\mathbf{r}_N - \mathbf{A}^N\mathbf{x}_0)$ between the reference and the unforced final states. The grammian must be invertible = system must be controllable.

2.2 Free final-state

The final state $\mathbf{x}_N$ can also be used as a parameter for our optimization. Hence $\mathrm{d}\mathbf{x}_N \neq 0$ and therefore

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1},\\
\boldsymbol{\lambda}_k &= \mathbf{Q}\mathbf{x}_k + \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1},\\
\mathbf{u}_k &= -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1},\\
\mathbf{S}_N\mathbf{x}_N &= \boldsymbol{\lambda}_N,\\
\mathbf{x}_0 &= \text{given}.
\end{aligned}$$

The state and the co-state at the final time are linearly related (but we know neither). A trick to proceed with the solution is to assume the linearity for all previous times—sweep method

$$\mathbf{S}_k\mathbf{x}_k = \boldsymbol{\lambda}_k$$

This will lead to *Difference Riccati Equation*

$$\mathbf{S}_k = \mathbf{Q} + \mathbf{A}^{\mathrm{T}}\mathbf{S}_{k+1}(\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S}_{k+1})^{-1}\mathbf{A}$$

Initialized with $\mathbf{S}_N$, it generates the sequence of matrices $\mathbf{S}_{N-1}, \mathbf{S}_{N-2}, \mathbf{S}_{N-3}, \ldots$

Optimal control is generated by linear time-varying state feedback (featuring Kalman gain)

$$\mathbf{u}_k = \underbrace{(\mathbf{I} + \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S}_{k+1}\mathbf{B})^{-1}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S}_{k+1}\mathbf{A}}_{\mathbf{K}_k}\,\mathbf{x}_k$$

3 LQ-optimal regulation over a infinite time interval

On a long enough (yet finite) interval, the steady-state values could be used to get a suboptimal control

$$\mathbf{K}_\infty \triangleq \lim_{k\to-\infty} \mathbf{K}_k \qquad \mathbf{S}_\infty \triangleq \lim_{k\to-\infty} \mathbf{S}_k$$

On an infinite interval this is actually optimal. Besides the iterative algorithm, the steady-state values can also be get by exploting

$$\mathbf{S}_\infty = \mathbf{S}_k = \mathbf{S}_{k+1}$$

yielding Discrete-time Algebraic Riccati Equation (DARE)

$$\mathbf{S}_\infty = \mathbf{A}^{\mathrm{T}}\left[\mathbf{S}_\infty - \mathbf{S}_\infty\mathbf{B}(\mathbf{B}^{\mathrm{T}}\mathbf{S}_\infty\mathbf{B} + \mathbf{R})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S}_\infty\right]\mathbf{A} + \mathbf{Q}$$

In the scalar (first-order) case, DARE is a quadratic equation (multiply both sides by the denominator)

$$s_\infty = a^2 s_\infty - \frac{a^2 b^2 s_\infty^2}{b^2 s_\infty + r} + q$$

3.1 Necessary and sufficient conditions of existence of a unique stabilizing solution

For $\mathbf{S} \geq 0$ (hence stabilizing $\mathbf{K}$)

- $(\mathbf{A}, \mathbf{B})$ is stabilizable
- $(\mathbf{A}, \sqrt{\mathbf{Q}})$ is detectable

If $\mathbf{S}$ is to be positive definite ($\mathbf{S} > 0$) (hence nonsingular), the system $(\mathbf{A}, \sqrt{\mathbf{Q}})$ must be observable.

4 Matlab commands

`dare`, `dlqr`

5 LQ-optimal tracking over a finite time

$$\begin{aligned}
\underset{\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{u}_0,\ldots,\mathbf{u}_{N-1}}{\text{minimize}} \quad & \frac{1}{2}(\mathbf{y}_N - \mathbf{r}_N)^{\mathrm{T}}\mathbf{S}_N(\mathbf{y}_N - \mathbf{r}_N)\\
& + \frac{1}{2}\sum_{k=0}^{N-1}\left[(\mathbf{y}_k - \mathbf{r}_k)^{\mathrm{T}}\mathbf{Q}(\mathbf{y}_k - \mathbf{r}_k) + \mathbf{u}_k^{\mathrm{T}}\mathbf{R}\mathbf{u}_k\right]\\
\text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \qquad \mathbf{x}_0 \text{ given},\\
& \mathbf{y}_k = \mathbf{C}\mathbf{x}_k\\
& \mathbf{S}_N \geq 0, \mathbf{Q} \geq 0, \mathbf{R} > 0.
\end{aligned}$$