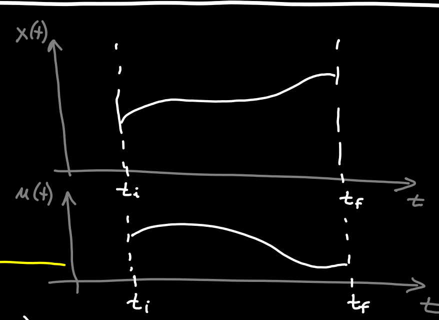
OPTIMAL CONTROL FOR CONTINUOUS-TIME SYSTEMS – CALCULUS OF VARIATIONS

$$\min_{u,x \in C^1[t_i,t_f]} \left[\phi(x(t_f), t_f) + \int_{t_i}^{t_f} L(x(t), u(t), t)\, dt \right]$$

$$\text{s.t.} \quad \dot{x}(t) = f(x(t), u(t), t), \qquad x(t_i) = r_i$$

$$h(x(t), u(t)) \leq 0 \qquad \nu(x(t_f), t_f) = 0 \quad \left(x(t_f) = r_f\right)$$

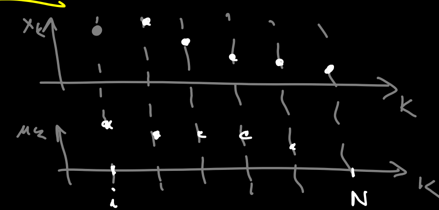


LQ-optimal control:

$$\min_{u,x} \left[\frac{1}{2} x^T(t_f) S_f x(t_f) + \frac{1}{2} \int_0^{t_f} \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) dt \right]$$

$$\text{s.t.} \quad \dot{x}(t) = A x(t) + B u(t), \qquad x(0) = r_0$$

compare with the discrete-time:



or infinite-horizon LQ:

$$\min \int_0^{\infty} \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) dt$$

$$\text{s.t.} \quad \dot{x}(t) = A x(t) + B u(t), \qquad x(0) = r_0$$

OPTIMIZATION OVER FUNCTIONS $\longrightarrow$ CALCULUS OF VARIATIONS

$$\boxed{\min_{y \in C^1[a,b]} J(y(x))}$$

function – indep. var. (posit.)

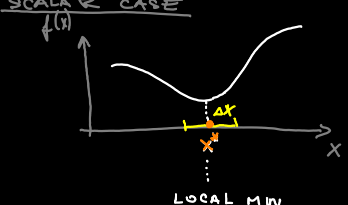
functional (= function of a function)

compare with

DIFFERENTIAL CALCULUS

$$\boxed{\min_{x \in \mathbb{R}} f(x)}$$

SCALAR CASE



$$f(\underbrace{x^* + \Delta x}_{x}) \geq f(x^*)$$

$$\Downarrow$$

$$\Delta f \geq 0 \quad \text{— differential}$$

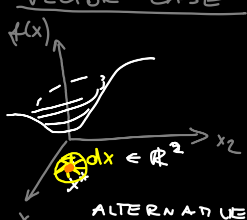
$$\Delta f = \underbrace{df}_{} + \mathcal{O}(\Delta x^2)$$

derivative.. $f' \cdot dx$

$$\Downarrow$$

$$df = 0 \implies f' = 0$$

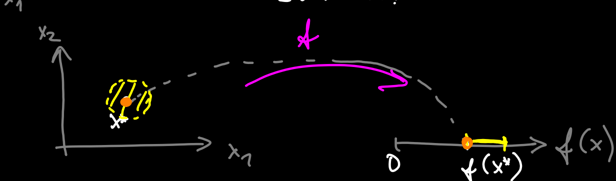
VECTOR CASE $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



$$df = \overbrace{(\nabla f)}^{\text{gradient}}{}^T \cdot dx$$

inner product

ALTERNATIVE VIEWPOINT:



$$y(x, \alpha) = \underbrace{y(x, 0)}_{y^*(x)} + \underbrace{\left.\frac{\partial y(x,\alpha)}{\partial \alpha}\right|_{\alpha=0} \cdot \alpha}_{} + \mathcal{O}(\alpha^2)$$

δy .. variation of function = differential w.r.t. α

$$\boxed{y = y^* + \delta y} \qquad \boxed{\delta y = \underbrace{\left.\frac{\partial y(x,\alpha)}{\partial \alpha}\right|_{\alpha=0}}_{\eta(x)} \cdot \alpha = \eta(x) \cdot \alpha}$$

$\eta(x) \in C^1$, α

$J(y+\delta y) - J(y)$

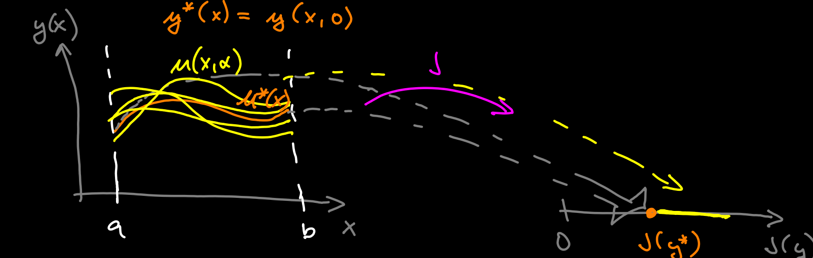
$$\boxed{\Delta J = \delta J + \mathcal{O}(\|y\|_2^2)}$$

$$\boxed{\delta J = \; ??? \; \delta y} = \overbrace{\int_{t_i}^{t_f} \frac{\delta J}{\delta y(x)} \cdot \delta y}^{\text{inner product}}$$

variation of functional

variational derivative

SCALAR Y



NEIGHBORHOOD ?

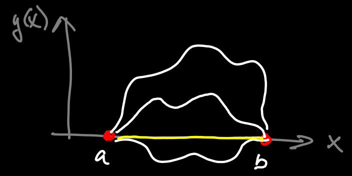
$$\frac{\delta J}{\delta y(x)} = ?$$

arbitrary but fixed

$$\frac{d J(y^* + \eta \cdot \alpha)}{d\alpha}$$

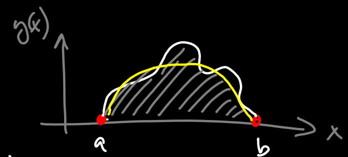
Examples

1.) Minimum distance between points



$$J = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = \int_a^b \sqrt{1 + y'} \, dx$$

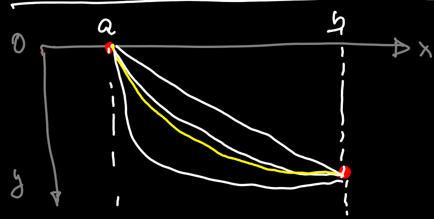
2.) Dido's problem — maximization of area (circumference fixed)



$$\max J, \quad J = \int_a^b y(x)\, dx$$

$$\text{s.t.} \quad \int_a^b \sqrt{1 - y'} \, dx = C$$

3.) Brachistochrone problem (shortest time)



$$\min J$$

$$J = \int_{t_i}^{t_f} dt = \int_{t_i}^{t_f} \frac{ds}{v} = \int_a^b \frac{\sqrt{1 - y'(x)}}{\sqrt{2 g y(x)}} \, dx$$

$$T + V = \frac{1}{2} m v^2 - m g y$$

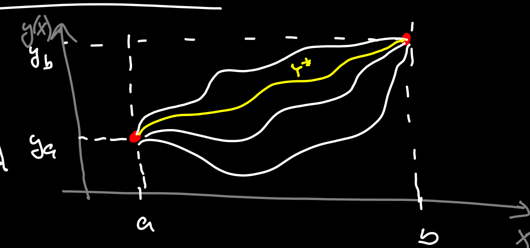
$$(T + V)_{t_i} = 0 \Rightarrow (T + V)_t = 0 \Rightarrow \frac{1}{2} \cancel{m} v^2 = \cancel{m} g y$$

$$v = \sqrt{2 g y}$$

$$\boxed{J(y(x)) = \int_a^b L(x, y(x), y'(x)) \, dx}$$

Basic problem of calculus of variations with fixed ends

$$\boxed{\min_{y(x)} \int_a^b L(x, y(x), y'(x)) \, dx, \quad y(a) = y_a, \quad y(b) = y_b}$$



$$y^*(x) \longrightarrow y(x)^* + \underbrace{\delta y(x)}_{\alpha \cdot \eta(x)}$$

$$\boxed{\delta J = \left.\frac{dJ}{d\alpha}\right|_{\alpha=0} \cdot \alpha}$$

$$\frac{d J(y^*(x) + \alpha \cdot \eta(x))}{d\alpha} = \frac{d}{d\alpha} \int_a^b L(x, \overset{y^* + \alpha \cdot \eta}{y(x)}, \overset{y^{*\prime} + \alpha \cdot \eta'}{y'(x)}) \, dx = \int_a^b \frac{d}{d\alpha} L(x, y^* + \alpha \cdot \eta, y^{*\prime} + \alpha \cdot \eta') \, dx = \int_a^b \left[\frac{\partial L}{\partial y} \eta + \frac{\partial L}{\partial y'} \cdot \eta'\right] dx$$

$$\left| \begin{matrix} u = \frac{\partial L}{\partial y'} & v' = \eta' \\ u' = \frac{d}{dx}\frac{\partial L}{\partial y'} & v = \eta \end{matrix} \right| = \underbrace{\cancel{\left[\frac{\partial L}{\partial y'} \eta\right]_a^b}}_{\eta(a) = \eta(b) = 0} + \underbrace{\int_a^b \overbrace{\left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right]}^{\frac{\delta J}{\delta y} \text{ ... variational derivative}} \eta(x) \, dx}_{= 0}$$

Fact: $\int_a^b f(x)\, \eta(x)\, dx = 0$ for arbitrary $\eta(x) \in C^1 \Longrightarrow f = 0$

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0}$$ ... EULER – LAGRANGE EQUATION

... second-order ODE

$$\boxed{L_y - \left(L_{y'x} + L_{y'y}\cdot y' + L_{y'y'}\cdot y''\right) = 0}$$

+ boundary conditions !

or

$$\boxed{\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}}$$

$$\delta J = \left.\frac{dJ}{d\alpha}\right|_{\alpha=0}\cdot\alpha = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right]\underbrace{\eta\alpha}_{\delta y}\,dx = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right]\delta y\,dx$$

But also

$$\boxed{\delta J = \delta\int_a^b L(x, y(x), y'(x))\,dx}$$

$$= \int_a^b \delta L(x, y(x), y'(x))\,dx = \int_a^b \left[\cancel{\frac{\partial L}{\partial x}\underbrace{\delta x}_{0}} + \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\delta y'\right]dx$$

$$= \int_a^b \left[\frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\underbrace{\delta y'}\right]dx = \int_a^b \left[\frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}(\delta y)'\right]dx$$

PER PARTES

$$\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$$

$$= \begin{vmatrix} u = \frac{\partial L}{\partial y'} & v' = (\delta y)' \\ u' = \frac{d}{dx}\frac{\partial L}{\partial y'} & v = \delta y \end{vmatrix} = \underbrace{\cancel{\left[\frac{\partial L}{\partial y'}\delta y\right]_a^b}}_{0} + \int_a^b \left[\frac{\partial L}{\partial y}\delta y - \frac{d}{dx}\frac{\partial L}{\partial y'}\delta y\right]dx$$

$$\boxed{\delta J = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right]\delta y\,dx}$$

SPECIAL CASES

1.) NO y :

$$0 = \frac{d}{dx}\frac{\partial L}{\partial y'} \;\Rightarrow\; \overset{p}{\frac{\partial L}{\partial y'}} = c_1 \quad \text{constant}$$

$$\boxed{p := \frac{\partial L}{\partial y'}}$$

2.) NO x :

$$L_y - \left(\cancel{L_{y'x}} + L_{y'y}\cdot y' + L_{y'y'}\cdot y''\right) = 0 \qquad /\cdot y'$$

$$L_y\cdot y' - L_{y'y}\cdot y'^2 - L_{y'y'}\,y''y' = 0$$

$$\frac{d}{dx}\left(L_{y'}\cdot y' - L\right) = 0$$

$$\Downarrow$$

$$L_{y'}\cdot y' - L = C_2 \quad \text{... constant}$$

$$\boxed{H := L_{y'}\cdot y' - L = \underbrace{p\cdot y' - L}}$$

$$H(x, y, y', p)$$

$$\left(L_{y'y}\cdot y' + L_{y'y'}\cdot y''\right)\cdot y' + L_{y'}\cdot y'' - L_y\cdot y' - L_{y'}\cdot y''$$
$$= L_{y'y}\cdot y'^2 + L_{y'y'}\cdot y''\cdot y' + \cancel{L_{y'}y''} - L_y\cdot y' - \cancel{L_{y'}\cdot y''}$$

Ex.: NO y ! switch from $x \to t$:

$$L = \frac{1}{2}m\dot y^2 + 0$$

$$L_{\dot y} = m\dot y =: p$$

NO x : $L = \frac{1}{2}m\dot y^2 - mg\cdot y$

$$H = \frac{1}{2}m\dot y^2 - \cancel{\frac{1}{2}m\dot y^2} + mgy \quad \text{... total en.}$$

Also:

$$\left.\begin{aligned} \dot{y} &= H_p \\ \dot{p} &= \frac{d}{dx} L_{\dot{y}} \underset{\substack{\uparrow \\ \text{E.L.}}}{=} L_y = -H_y \end{aligned}\right\} \quad \boxed{\begin{bmatrix} \dot{y} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} H_p \\ -H_y \end{bmatrix}}$$

Hamilton's canon. eq.

And yet another property:

$$\frac{\partial H_y}{\partial \dot{y}} = p - \frac{\partial L}{\partial \dot{y}} = 0 \implies$$ H has a stationary point wrt. the third variable

$$H(x, y, \underline{z}, p) \longrightarrow \boxed{\left.\frac{\partial H(x, y^*, z, p^*)}{\partial z}\right|_{z=\dot{y}^*} = 0}$$

SECOND-ORDER CONDITIONS

- necessary $\quad L_{\dot{y}\dot{y}} \geq 0$
- sufficient $\quad L_{\dot{y}\dot{y}} > 0 \quad$ ... not enough (should be on a short interval)

$$\Downarrow$$

$$H_{\dot{y}\dot{y}} = -L_{\dot{y}\dot{y}} \qquad H = \underbrace{L_{\dot{y}}}_{p} \cdot \dot{y} - L$$

$$\Downarrow$$

$$\boxed{H_{\dot{y}\dot{y}} < 0} \implies$$ H maximized at extremal $H_{\dot{y}} = 0$

$$\boxed{H(x, y^*, \dot{y}^*, p^*) \geq H(x, y^*, z, p^*)}$$

EQUALITY CONSTRAINTS

$F(x, y, \dot{y}) = 0$ ... implicit ODE

$$J^{AUG} = \int_a^b L(x, y, \dot{y})\, dx + \int_a^b \lambda(t) \cdot F(x, y, \dot{y})\, dx = \int_a^b \underbrace{\left[L + \lambda F\right]}_{L^{AUG}(x, y, \dot{y}, \lambda)} dx$$

Explicit ODE: $\dot{y} = f(x, y(x), u(x))$

$$J = \int_a^b L(x, y, \dot{y})\, dx = \int_a^b \tilde{L}(x, y, u)\, dx$$

Ex: $\dot{y} = u$: $\quad J = \int_a^b L(x, y, \underline{u})\, dx$

OPTIMAL CONTROL - FIXED FINAL STATE, FINITE HOR.

$$\min \int_{t_i}^{t_f} L(t, x(t), u(t))\, dt \qquad ! \quad \begin{aligned} x &\to t \\ y(x) &\to x(t) \end{aligned}$$

either

s.t. $\quad \dot{x}(t) = f(x(t), u(t), t), \quad x(t_i) = x_i, \quad x(t_f) = r_f$

$p \to \lambda$

OR

$$J^{AUG}(t, x, u, \lambda) = \int_{t_i}^{t_f} \underbrace{\left[L(x, u, t) + \lambda^T(t) \cdot \left(\dot{x}(t) - f(x, u, t) \right) \right]}_{L^{AUG}(t,\, x(t),\, \dot{x}(t),\, u(t),\, \lambda(t))} dt$$

vs

$$L(t, x, \dot{x})$$

$$\begin{bmatrix} x \\ u \\ \lambda \end{bmatrix} \quad \begin{bmatrix} \dot{x} \\ \dot{u} \\ \dot{\lambda} \end{bmatrix}$$

E.L. wrt $\begin{bmatrix} x \\ u \\ \lambda \end{bmatrix}$

$$\nabla_x L^{AUG} = \frac{d}{dt} \nabla_{\dot{x}} L^{AUG}$$

$$\nabla_u L^{AUG} = \cancel{\frac{d}{dt} \nabla_{\dot{u}} L^{AUG}} = 0$$

$$\nabla_\lambda L^{AUG} = 0 \implies$$

$$\boxed{\begin{aligned} \nabla_x L - \nabla_x f \cdot \lambda &= \dot{\lambda} \\ \nabla_u L - \nabla_u f\, \lambda &= 0 \\ \dot{x} - f(x, u, t) &= 0 \end{aligned}}$$

Ex.: LQ-optimal, fixed final state: $L = \frac{1}{2}\left[x^T Q x + u^T R u \right]$

$$Qx - A^T \lambda = \dot{\lambda}$$

$$\boxed{u = R^{-1} B^T \lambda} \leftarrow Ru - B^T \lambda = 0$$

$$Ax + Bu = \dot{x}$$

$$\boxed{\begin{aligned} \dot{x} &= Ax + B R^{-1} B^T \lambda \\ \dot{\lambda} &= Qx - A^T \lambda \end{aligned}} \Bigg\} \begin{array}{l} \text{TWO-POINT BVP} \\ 2n \text{ eq's} \\ 2n \text{ BC's} \end{array}$$

NEEDED: $x(t_i) = r_i$, $x(t_f) = r_f$

$\boxed{\text{MATLAB: BVP4C}}$

Using Hamiltonian $H(t, x, u, \lambda) = \lambda^T \cdot \dot{x} - L$

LQ: $H = \lambda^T \dot{x} - \frac{1}{2}\left[x^T Q x + u^T R u \right]$

Ham. can. eqs

$$\boxed{\begin{aligned} \dot{x} &= \nabla_\lambda H \\ \dot{\lambda} &= -\nabla_x H \\ \hline 0 &= \nabla_u H \end{aligned}}$$

grad. based ... $u(t)$

$\uparrow$

$\lambda(t)$

$\uparrow$

$\lambda(t_f)$

$\uparrow$

Solving LQ - fixed final state

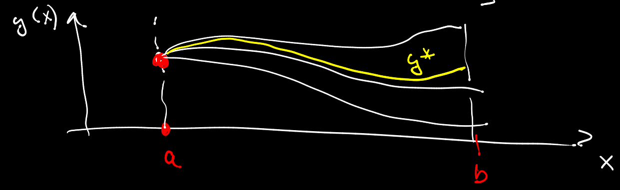
assume $Q = 0 \implies J = \int_0^{t_f} u^T R u \, dt$

$$\dot{x} = Ax + B R^{-1} B^T \lambda \to x(t) = \ldots x(t_f)$$

$$\dot{\lambda} = \quad - A^T \lambda \to \lambda(t) = \ldots \lambda(t_f)$$

FREE FINAL STATE ...

back to CALCULUS OF VARIATIONS



$$\underbrace{\left[\frac{\partial L}{\partial \dot y}\,\delta y\right]_a^b}\; + \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial \dot y}\right]\delta y \; dx$$

$$\underbrace{\left.\frac{\partial L}{\partial \dot y}\,\delta y\right|_b + \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial \dot y}\right]\delta y \; dx = 0}$$

$$\Downarrow$$

$$\boxed{\begin{array}{c} \dfrac{\partial L}{\partial y} - \dfrac{d}{dx}\dfrac{\partial L}{\partial \dot y} = 0 \\ \boxed{\left.\dfrac{\partial L}{\partial \dot y}\right|_b = 0} \end{array}}$$

$\downarrow$ in control :

$$\boxed{\left.\frac{\partial L^{AUG}}{\partial \dot x}\right|_{t_f} = \lambda(t_f) = 0}$$

$$J = \underbrace{\phi(x_{t_f})} + \int_{t_i}^{t_f} L \, dt$$

$$\downarrow$$

$$\int_{t_i}^{t_f} \underbrace{\frac{d}{dt}\phi(x,t)}_{\cancel{\frac{\partial \phi}{\partial t}} + \frac{\partial \phi}{\partial x}\dot x} dt + \cancel{\phi(x(t_i))}$$

TIME INVARIANT

$$J = \int_{t_i}^{t_f} \left[L + \frac{\partial \phi}{\partial x}\dot x + \lambda^T\left(\dot x - f(x,u,t)\right)\right] dt$$

$$\lambda(t_f) + \frac{\partial \phi}{\partial x}(t_f) = 0$$

$$\Downarrow$$

LQ: $\phi = \frac{1}{2} x^T(t_f)\, S\, x(t_f)$

$$\Downarrow$$

$$\boxed{\lambda(t_f) + S \cdot x(t_f)}$$

LQ :

$$\dot x = Ax + Bu$$

$$\dot \lambda = Qx - A^T \lambda$$

$$u = -R^{-1} B^T \lambda$$

BC: $x(t_i) = x_i$

$$S\,x(t_f) = -\lambda(t_f)$$

$$\Downarrow$$

$$S\,x(t) = -\lambda(t)$$

$$\dot{S}x + S\dot{x} = -\dot{\lambda}$$

$$\dot{S}x + S(Ax + B\underset{\substack{\uparrow \\ -R^{-1}B^TSx}}{u}) = -(Qx - A^T\underset{\substack{\uparrow \\ -Sx}}{\lambda})$$

$$-\dot{S}\cancel{x} = SA\cancel{x} - SBR^{-1}B^TS\cancel{x} + Q\cancel{x} + A^TS\cancel{x}$$

$$\boxed{-\dot{S} = SA + A^TS + Q - SBR^{-1}B^TS}$$

dif. RE

$\dot{S} = 0 \Rightarrow$ ARE