Output LQR, LQG and $\mathcal{H}_2$ optimal control

Graduate course on Optimal and Robust Control (spring'20)

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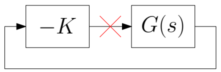
Lecture outline

Guaranteed stability margins for LQ optimal regulator



$$\mathbf{G}(s) = (sI - \mathbf{A})^{-1}\mathbf{B}$$

Open-loop transfer function

$$\mathbf{L}(s) = \mathbf{K}\mathbf{G}(s) = \mathbf{K}(sI - \mathbf{A})^{-1}\mathbf{B}$$

First, single-input system (L is scalar):
Return difference $1 + L(s)$ on the imaginary axis
Magnitude $|1 + L(j\omega)| = |-1 - L(j\omega)| \quad \forall \omega \in \mathbb{R}$.
Easier to evaluate

$$|1 + L(j\omega)|^2 = \overline{(1 + L(j\omega))}(1 + L(j\omega))$$

Conjugate system

$$L^*(s) = L(-s) \qquad \text{some use } \tilde{L}(s)$$

$$\boxed{L^*(j\omega) = \overline{L(j\omega)}}$$

For MIMO systems

$$\mathbf{L}^*(s) = \mathbf{L}^\mathsf{T}(-s)$$

$$\boxed{\mathbf{L}^*(j\omega) = \left(\overline{\mathbf{L}(j\omega)}\right)^\mathsf{T}}$$

Exactly what Matlab does with **L'**

Kalman's identity for return difference for LQR

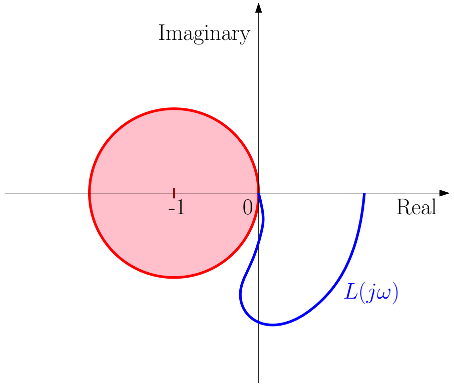
For $\mathbf{R} = \rho I$

$$\boxed{(\mathbf{I} + \mathbf{L}^*)\,(\mathbf{I} + \mathbf{L}) = \mathbf{I} + \frac{1}{\rho}\mathbf{B}^{\mathsf{T}}(-sI - \mathbf{A})^{-\mathsf{T}}\mathbf{Q}(sI - \mathbf{A})^{-1}\mathbf{B} \geq \mathbf{I}}$$

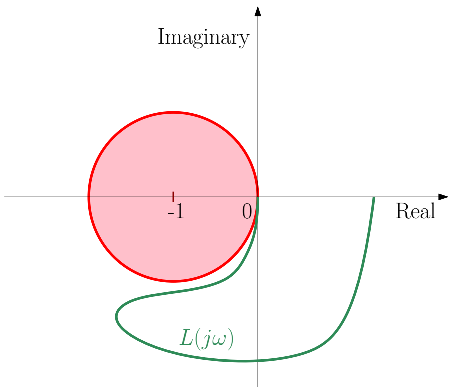
For SISO systems

$$\boxed{|1 + KG(j\omega)| \geq 1, \quad \forall\omega}$$

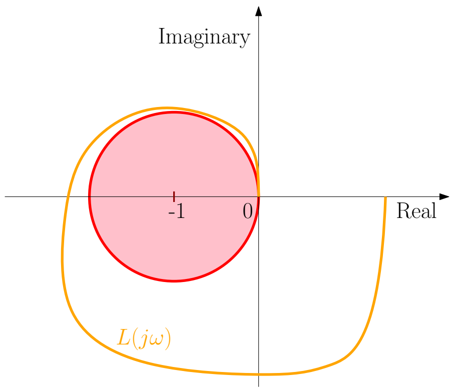
Stability margins



Stability margins



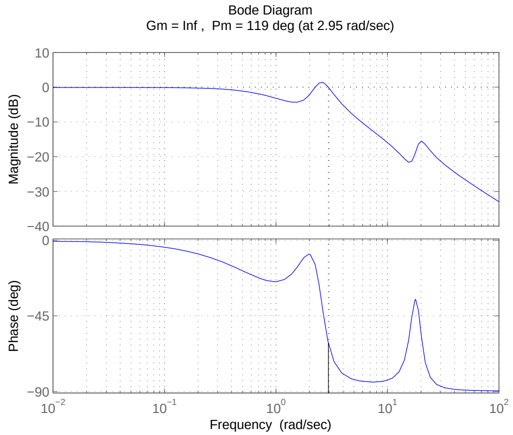
Stability margins



$$\boxed{GM_+ = \infty, \quad GM_- = \frac{1}{2}, \quad PM_- \leq -60^\circ, \quad PM_+ \geq 60^\circ}$$

Run a few times in Matlab

```
n = 5;                     % order of the plant
G = rss(n,n,1);            % random system, 1 input, n outputs
G.c = eye(n,n);            % all state variables as outputs
G.d = zeros(n,1);          % zero feedthrough
Q = eye(n,n);              % penalty on x
R = 1;                     % penalty on u
[K,S,e] = lqr(G,Q,R)       % LQR design
L = K*G;                   % open-loop transfer function
margin(L)
```



Stochastic LQ optimal regulator

Plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_w\mathbf{w}(t), \qquad \mathbf{x}(0) = \ldots$$

with random initial conditions and random disturbance.
Random initial states: zero mean and a covariance matrix

$$\mathbf{E}\left\{\mathbf{x}(0)\right\} = \mathbf{0},$$
$$\mathbf{E}\left\{\mathbf{x}(0)\mathbf{x}^{\mathrm{T}}(0)\right\} = \mathbf{\Sigma}_{\mathbf{x}}(0)$$

Disturbace: white noise with spectral density $\mathbf{S}_w$ and uncorellated with the initial state.

$$\mathbf{E}\left\{\mathbf{w}(t)\right\} = \mathbf{0},$$
$$\mathbf{E}\left\{\mathbf{w}(t)\mathbf{w}^{\mathrm{T}}(t+\tau)\right\} = \mathbf{S}_w\delta(\tau)$$

The cost function

$$J = \frac{1}{2}\mathbf{E}\left\{\mathbf{x}^{\mathrm{T}}(t_{\mathrm{f}})\mathbf{S}\mathbf{x}(t_{\mathrm{f}}) + \int_0^{t_{\mathrm{f}}}\left[\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)\right]dt\right\}$$

But what if $t_{\mathrm{f}} = \infty$?

$$J = \frac{1}{2}\mathbf{E}\left\{\int_0^{\infty}\left[\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)\right]dt\right\}$$

does not (generally) converge.
Scaling by $\frac{2}{t_{\mathrm{f}}}$

$$J' = \frac{1}{2}\mathbf{E}\left\{\lim_{t_{\mathrm{f}}\to\infty}\frac{2}{t_{\mathrm{f}}}\int_0^{t_{\mathrm{f}}}\left[\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)\right]dt\right\}$$
$$= \mathbf{E}\left\{\mathbf{x}^{\mathrm{T}}(\infty)\mathbf{Q}\mathbf{x}(\infty) + \mathbf{u}^{\mathrm{T}}(\infty)\mathbf{R}\mathbf{u}(\infty)\right\}$$

Stochastic LQR is identical to the deterministic one

1. optimal controller must necessarily be a feedback controller because of the random initial states and disturbances
2. if Gaussian white noise disturbances assumed, the optimal controller is a linear state feedback
3. controller **independent** of the initial state covariance matrix and the disturbance spectral density matrix.
4. Solution via (algebraic) Riccati equation exactly as in the deterministic case.

The steady-state covariance matrix for the state vector can be found by solving the Lyapunov equation

$$\boxed{(\mathbf{A}-\mathbf{B}\mathbf{K})\mathbf{\Sigma}_x(\infty) + \mathbf{\Sigma}_x(\infty)(\mathbf{A}-\mathbf{B}\mathbf{K})^{\mathrm{T}} + \mathbf{B}_w\mathbf{S}_w\mathbf{B}_w^{\mathrm{T}} = 0}$$

Ex.: Stochastic LQR for a satellite tracking antenna

Pointing antenna subject to random wind torque

$$\begin{bmatrix}\dot\theta(t)\\ \ddot\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -0.1\end{bmatrix}\underbrace{\begin{bmatrix}\theta(t)\\ \dot\theta(t)\end{bmatrix}}_{\mathbf{x}} + \begin{bmatrix}0\\ 0.001\end{bmatrix}u(t) + \begin{bmatrix}0\\ 0.001\end{bmatrix}w(t)$$

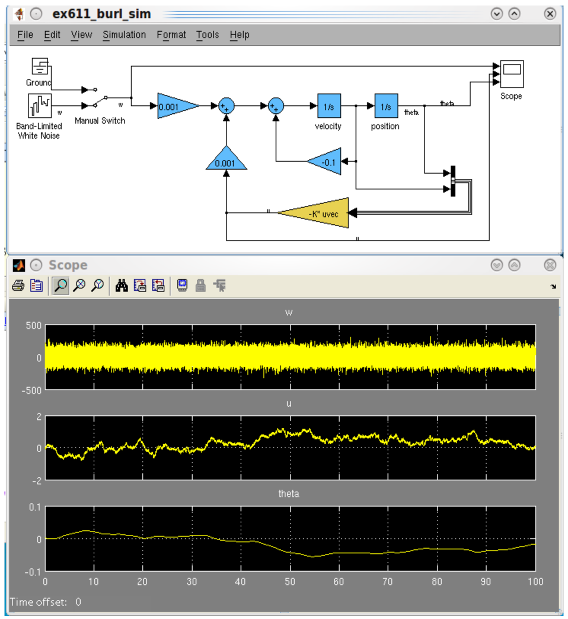
where $\theta(t)$ is a pointing error $[deg]$, $u(t)$ is a control torque $[Nm]$ and $w(t)$ is a random wind toque $[Nm]$.
The wind torque is modelled as white noise with a spectral density $S_w = 5000N^2m^2/Hz$.
Cost function

$$J = \mathbf{E}\left[\begin{bmatrix}\theta(\infty) & \dot\theta(\infty)\end{bmatrix}\begin{bmatrix}180 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\theta(\infty)\\ \dot\theta(\infty)\end{bmatrix} + u^2(\infty)\right]$$

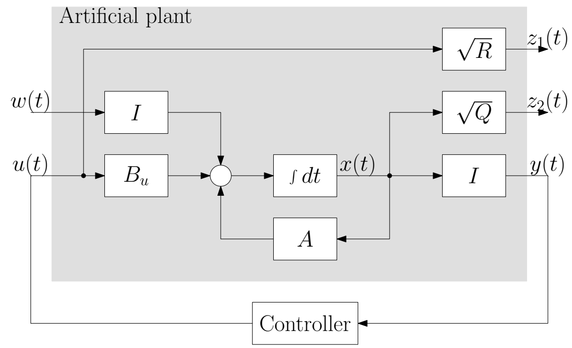
LQR in Matlab

```
A = [0 1; 0 -0.1];
Bu = [0; 0.001];
Bw = [0; 0.001];

Sw = 5000;

Q = [180 0; 0 0];
R = 1;

[S,D,K] = care(A,Bu,Q,R);

Sx = lyap(A-Bu*K,Bw*Sw*Bw')
```

$$\mathbf{\Sigma}_x(\infty) = \begin{bmatrix}0.9854\text{deg}^2 & 0\\ 0 & 0.1141\frac{\text{deg}^2}{s^2}\end{bmatrix}$$



Alternative input-output interpretation of stochastic LQR



$$J = \mathbf{E}\left\{\begin{bmatrix}\mathbf{z}_1^T(\infty) & \mathbf{z}_2^T(\infty)\end{bmatrix}\begin{bmatrix}\mathbf{z}_1(\infty)\\ \mathbf{z}_2(\infty)\end{bmatrix}\right\}$$

$\mathcal{H}_2$ system norm

For a stable and strictly proper LTI system G

$$\boxed{\|\mathbf{G}\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left[\mathbf{G}^*(j\omega)\mathbf{G}(j\omega)\right] d\omega}}$$

By using Parseval's theorem (relating inner products in time and frequency domains)

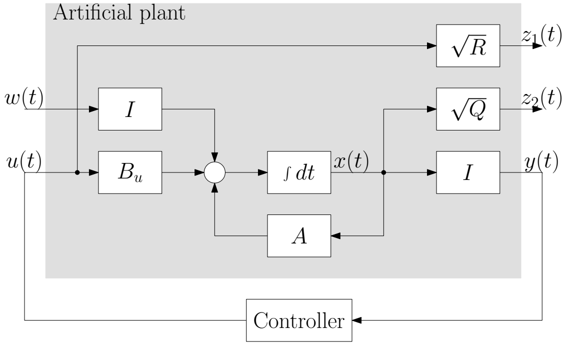
$$\boxed{\|\mathbf{G}\|_2 = \sqrt{\int_{0}^{\infty} \operatorname{tr}\left[\mathbf{g}^{\mathsf{T}}(t)\mathbf{g}(t)\right] dt}}$$

$\mathcal{H}_2$ norm as a gain of a system subject to a stationary white noise input

$$\boxed{\mathbf{E}\left\{\mathbf{y}^{\mathsf{T}}(\infty)\mathbf{y}(\infty)\right\} = \|\mathbf{G}\|_2^2\, S_w}$$

See, for example, Doyle, Francis, Tannenbaum (1990) at `https://www.control.utoronto.ca/people/profs/francis/dft.html`.

$\mathcal{H}_2$-optimal control



Find a stabilizing controller that minimizes $\mathcal{H}_2$ norm of the closed loop system.
Matlab: **h2syn()** from Robust Control Toolbox
Can easily extend to output feedback control, nonwhite noise, ...

Output feedback LQR

Not all the states are measured:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}(+\mathbf{D}\mathbf{u}).$$

Find the optimal output feedback

$$\mathbf{u} = -\mathbf{K}\mathbf{y}$$

that stabilizes the system and minimizes

$$J = \frac{1}{2}\int_0^{\infty}\left(\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathsf{T}}\mathbf{R}\mathbf{u}\right) dt$$

Generally nonconvex numerical optimization.

Optimal LQG output feedback control - state feedback and estimator

Combination of LQ-optimal state feedback and optimal estimator (Kalman filter).

Estimator (observer) of the states

Plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}$$

Observer (estimator)

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \underbrace{\mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}})}_{\text{correction}}$$

$$\boxed{\dot{\hat{\mathbf{x}}} = \underbrace{(\mathbf{A} - \mathbf{L}\mathbf{C})}_{\mathbf{A}_o}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{y}}$$

Kalman filter

Plant

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_w\mathbf{w}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t) \end{aligned}$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are white noises with spectral densities $\mathbf{S}_w$ and $\mathbf{S}_v$, respectively.
Observer design is **dual** to the state feedback design (ARE needs to be solved)

$$\begin{aligned} \mathbf{L} &= \mathbf{\Sigma}_e(\infty)\mathbf{C}^{\mathsf{T}}\mathbf{S}_v^{-1} \\ \mathbf{0} &= \mathbf{A}\mathbf{\Sigma}_e(\infty) + \mathbf{\Sigma}_e(\infty)\mathbf{A}^{\mathsf{T}} + \mathbf{B}_w\mathbf{S}_w\mathbf{B}_w^{\mathsf{T}} - \mathbf{\Sigma}_e(\infty)\mathbf{C}^{\mathsf{T}}\mathbf{S}_v^{-1}\mathbf{C}\mathbf{\Sigma}_e(\infty) \end{aligned}$$

where

$$\mathbf{\Sigma}_e(t) = \mathbf{E}\left\{[\mathbf{x}(t) - \hat{\mathbf{x}}(t)][\mathbf{x}(t) - \hat{\mathbf{x}}(t)]^{\mathsf{T}}\right\}$$

Matlab: **kalman()**

Example: Kalman filter

Estimate the range and radial velocity of an aircraft from noisy radar measurements.
Model

$$\begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

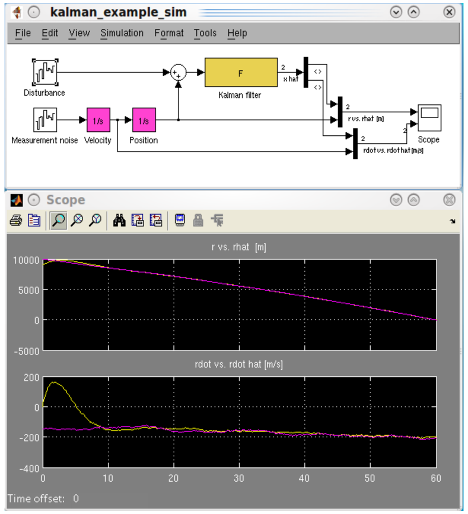
where $r(t)$ is the actual range of the aircraft. The range measurements are given

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + v(t)$$

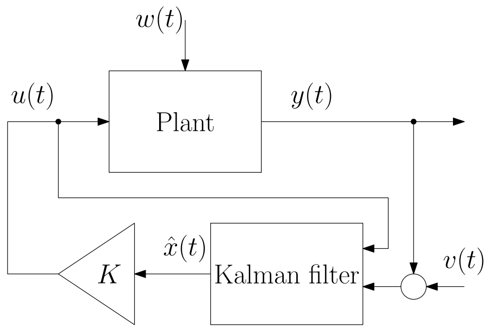
Initial conditions

$$\begin{bmatrix} r(0) \\ \dot{r}(0) \end{bmatrix} = \begin{bmatrix} 10000m \\ -150m/s \end{bmatrix}$$

$$S_w = 4 \frac{m^2}{s^4 Hz}$$
$$S_v = 10^4 \frac{m^2}{Hz}$$
$$\Sigma_e(0) = \begin{bmatrix} 10^6 m^2 & 0 \\ 0 & 4 \times 10^5 m^2/s^2 \end{bmatrix}$$



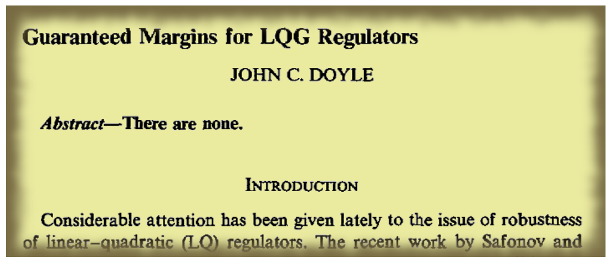
Combined Kalman filtering and LQ optimal control: LQG optimal control



Numerical solution: 2 AREs
Matlab: **lqgreg()** combines the results of **lqr()** (or **dlqr()**) and **kalman()**

Example: satellite tracking antenna with noisy measurements of angle

Stability margins of LQG


Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

***Abstract*—There are none.**

INTRODUCTION

Considerable attention has been given lately to the issue of robustness of linear-quadratic (LQ) regulators. The recent work by Safonov and

J. Doyle, "Guaranteed margins for LQG regulators," IEEE Transactions on Automatic Control, vol. 23, no. 4, pp. 756–757, Aug. 1978, doi: 10.1109/TAC.1978.1101812.

Example [Doyle 1978]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t),$$

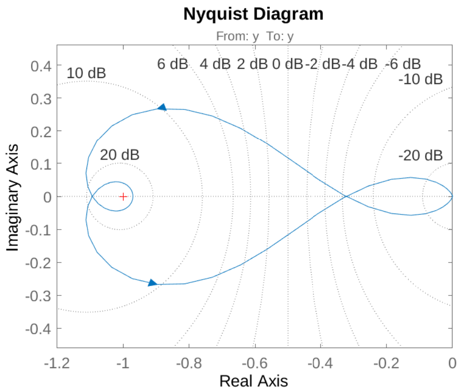
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) + v(t)$$

$$\mathbf{Q} = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix},$$

$$R = 1,$$

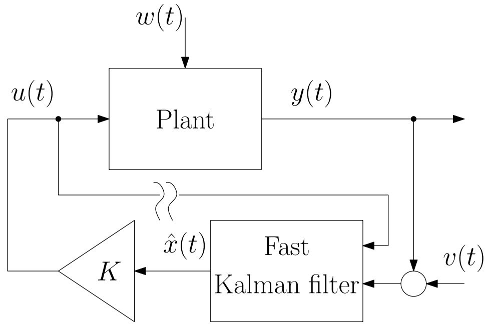
$$S_w = 10,$$

$$S_v = 1$$



LTR control - Heuristic way to improve robustness of LQG

Make it similar to LQR $\Longrightarrow$ make the Kalman filter rely less on u



But how?

Introduce fictitious noise $w_f(t)$ during the design (not in reality) that enters the system in the same way as the control signal

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_w\mathbf{w}(t) + \mathbf{B}_u\mathbf{w}_f(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \begin{bmatrix} \mathbf{B}_w & \mathbf{B}_u \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{w}_f(t) \end{bmatrix} \end{aligned}$$

Spectral density of the white noise disturbance is then

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_w & 0 \\ 0 & S_{w_f} I \end{bmatrix}$$

Ex.: LQG/LTR for a robot arm

Single link robotic arm is to be held vertical

$$J\ddot{\theta}(t) = mg \sin\theta(t) + u(t)$$

Linearizing and submitting some numbers yields

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} w$$

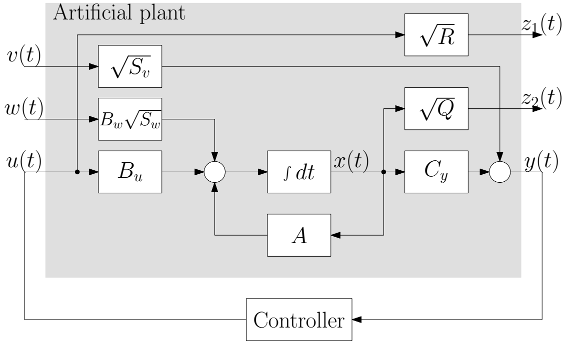
where w is a random disturbance by a torquer circuit with a spectral density $S_w = 1$. The angular position of the arm is measured

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + v$$

where the measurement noise spectral density is $S_v = 1$. Steady state LQG controller should minimizes

$$J = \mathbf{E}\left\{\theta^2 + 16u^2\right\}$$

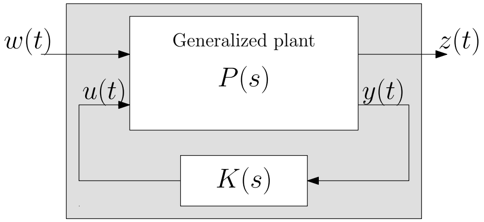
LQG as $\mathcal{H}_2$-optimal control



Generalized plant

- two types of inputs:
  - exogenous inputs: random disturbance $w(t)$ and noise $v(t)$
  - control inputs: control command $u(t)$
- two types of outputs:
  - regulated outputs: $z_1(t)$ and $z_2(t)$ to be beaten to zero
  - measured outputs: $y(t)$

$\mathcal{H}_2$-optimal control in full generality



$$\boxed{\underset{K(s)\text{ stabilizing}}{\text{minimize}} \|N(P,K)\|_2}$$

To conclude the LQ story

1. design usable controllers by minimizing the integral of some functions of the states and the inputs
2. minimization of an LQ integral cost can be reformulated as H2 system norm minization

How about using other system norms?