Intro to robustness and modeling uncertainty in dynamical systems

Graduate course on Optimal and Robust Control

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Lecture outline

Uncertainty and robustness

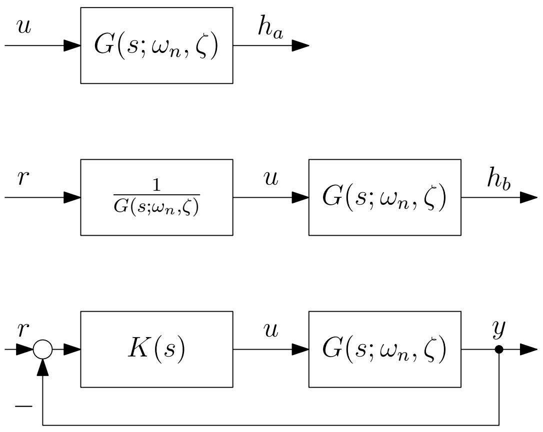
Definition (Uncertainty)
Deviation of the mathematical model from the reality.

Definition (Robustness)
Insensitivity of control system to uncertainty.

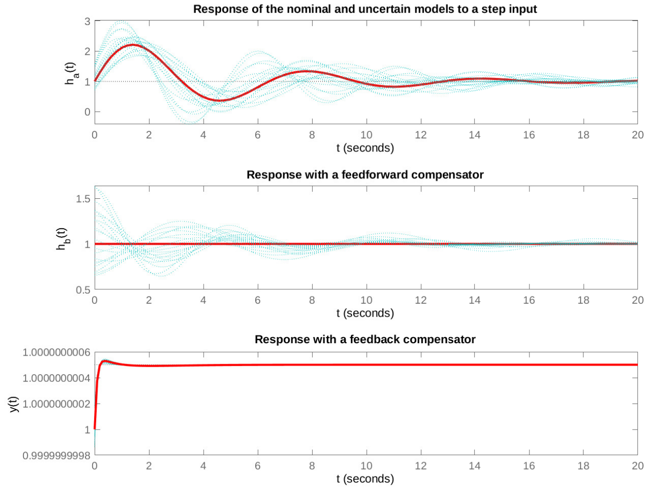
Where does the uncertainty come from?

- physical parameters are not known exactly ($\pm 5\%$)
- physical parameters can vary in time
- nonlinear systems linearized around a given operating point working around another operating point
- imperfect understanding or even misunderstanding the physics of the problem ($\omega \uparrow$)
- intentionally using simpler model (do not need PhD, $t_{\text{modelling}} = \$$)
- intentionally using simpler model to cut the computational cost ($t_{computation} = \$$)

Robustness as one of 3 reasons for feedback



Robustness as one of 3 reasons for feedback



Some robustness provided also by an integrator

The error in steady state is zero for whatever parameters of the systems as long as the closed-loop system is stable.

$$G(s) = \frac{k}{Ts+1}, \quad K(s) = \frac{Ps+I}{s}$$

The transfer function from the disturbance acting at the output to the regulation error is

$$S = \frac{1}{1+GK} = \frac{Ts^2+s}{Ts^2+(kP+1)s+kI}$$

Limit theorem about the gain in steady state

$$\lim_{s\to 0} S(s) = 0$$

Classification of models of uncertainty

- real parametric uncertainty
  - interval uncertainty (physical parameters at intervals)
  - multiple models and polytopic uncertainty (set of operating points)
- dynamic (also frequency-domain or norm-bounded) uncertainty
  - unstructured uncertainty
  - structured uncertainty (several unstructured unc.)
- stochastic models
  - stochastic disturbance

Uncertainty in real (physical) parameters

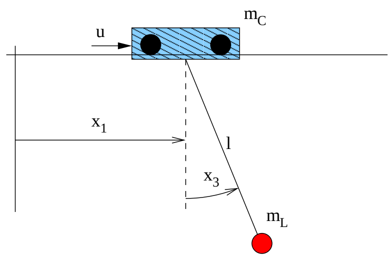
Advantages

- simple interpretation: value ±5% or in an interval.

Disadvantages

- unknown parameters are usually not the only trouble in modeling the system
- methods for control design not as mature as those for frequency based uncertainty models

Real parametric uncertainty



$$
\begin{aligned}
m_L &\in [m_L^{\min}, m_L^{\max}] \\
\ell &\in [\ell^{\min}, \ell^{\max}]
\end{aligned}
$$

$$
\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_L}{m_C} g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(m_L+m_C)g}{m_C \ell} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m_C} \\ 0 \\ -\frac{1}{m_C \ell} \end{bmatrix} u(t)
$$

Entering parametric uncertainties in Matlab

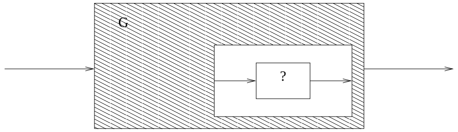
- Control Systems Tbx: array of LTI objects
- Robust Control Tbx: class UREAL, USS.
- Polynomial Tbx: some functions for analysis

Literature on parametric uncertainties

1. S. P. Bhattacharyya, H. Chapellat, L. H. Keel, and L. H. Keel, Robust Control: The Parametric Approach, Prentice Hall, 1995.
2. J. Ackerman, A. Bartlett, and D. Kaesbauer, Robust Control: Systems With Uncertain Physical Parameters, Springer-Verlag, 1993.
3. Ross. B. Barmish, New Tools for Robustness of Linear Systems, Macmillan Coll Div, 1993.

Unstructured dynamic uncertainty

Not only parameters but even system order, time delay, phase is uncertain. The system contains uncertain dynamics.



Simplest transfer function model of uncertainty is $\Delta(s)$

$$\max_{\omega} |\Delta(j\omega)| \leq 1, \quad \Delta \text{ stable}$$

But typically uncertainty higher at higher frequencies—shaping of the frequency characteristics using a some function $w(\omega)$, for computational reasons approximable using a low-order rational transfer function $W(s)$ ($W(j\omega) \approx w(\omega)$ on the imaginary axis)
The transfer function model of uncertainty then

$$\boxed{W(s)\,\Delta(s), \quad \max_{\omega} |\Delta(j\omega)| \leq 1, \quad \Delta \text{ stable}}$$

$\mathcal{H}_\infty$ norm of a system (model)

For SISO systems

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} |G(j\omega)|$$

For MIMO systems

$$\|\mathrm{G}\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(\mathrm{G}(j\omega))$$

where $\bar{\sigma}$ is the largest singular value.
Matlab: CST: **norm(G,Inf)**, RCT: **hinfnorm(G)**

Singular values, singular value decomposition (SVD)

$$\mathrm{M} = \mathrm{U} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \sigma_n \end{bmatrix}}_{\Sigma} \mathrm{V}^*$$

$$\mathrm{V}\mathrm{V}^* = \mathrm{I} = \mathrm{V}^*\mathrm{V}$$

$$\mathrm{U}\mathrm{U}^* = \mathrm{I} = \mathrm{U}^*\mathrm{U}$$

$$\sigma_i \in \mathbb{R}_+, \forall i$$

$$\bar{\sigma} = \sigma_1$$

Matlab: **svd**

Input-output interpretation of $\mathcal{H}_\infty$ norm

Viewing the dynamic system G as an operator mapping input *bounded-energy* signals into output *bounded-energy* signals

$$G: \ \mathcal{L}_2 \longrightarrow \mathcal{L}_2$$

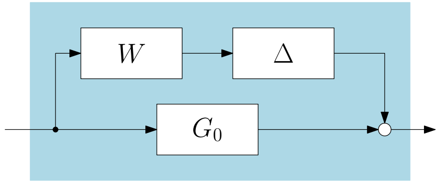
the norm describes the worst-case *energy gain* of the system

$$\boxed{\|G(s)\|_\infty = \sup_{u(t) \in \mathcal{L}_2 \setminus 0} \frac{\|y(t)\|_2}{\|u(t)\|_2}}$$

Scaling necessary to get any useful info from MIMO models! See Skogestad's book, section 1.4, pages 5–8.
`https://folk.ntnu.no/skoge/book/ps/book1-3.pdf`

Models of uncertain systems—additive

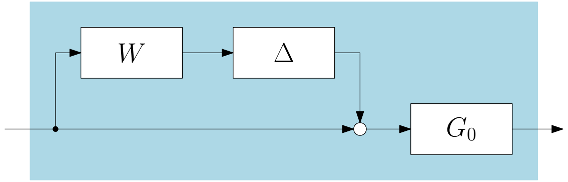
$$G(s) = \underbrace{G_0(s)}_{\text{nominal model}} + \underbrace{W(s)\Delta(s)}_{\text{additive uncertainty}}$$



$$|G(j\omega) - G_0(j\omega)| < |W(j\omega)|$$

Models of uncertain systems—(input-)multiplicative

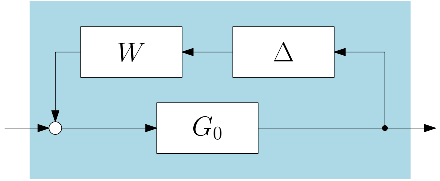
$$G(s) = (1 + W(s)\Delta(s))G_0(s)$$



$$\frac{|G(j\omega) - G_0(j\omega)|}{|G_0(j\omega)|} < |W(j\omega)|$$

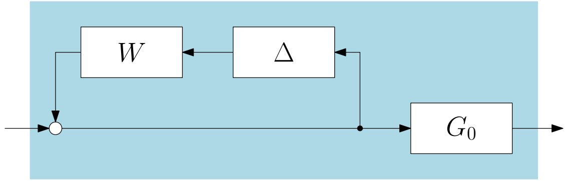
- Always some uncertainty at the input. Example with flow control with the measurement accuracy $\pm 1\%$; step $1 \to 1.1$ l/min, could be actually $0.99 \to 1.01$ l/min, rel. error is $0.02/0.1 = 20\%$.

Models of uncertain systems—inverse additive



Enable to describe uncertain unstable dynamics.

Models of uncertain systems—inverse multiplicative



Enable to describe uncertain unstable dynamics.

Linear Fractional Transformation (LFT)

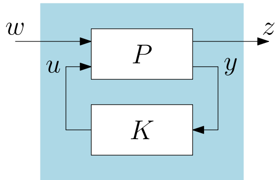
For matrices P and N sized $(n_1 + n_2) \times (m_1 + m_2)$ and divided into blocks like

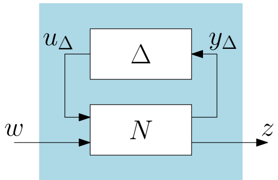
$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and matrices Δ a K sized $m_1 \times n_1$ and $m_2 \times n_2$, **lower** and **upper LFT** is

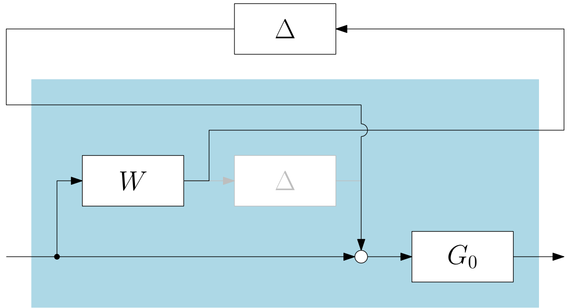
$$\boxed{\mathcal{F}_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}}$$

$$\boxed{\mathcal{F}_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}}$$





LFT continued: input-multiplicative uncertainty as LFT



$$P = \begin{bmatrix} 0 & W \\ G_0 & G_0 \end{bmatrix}$$

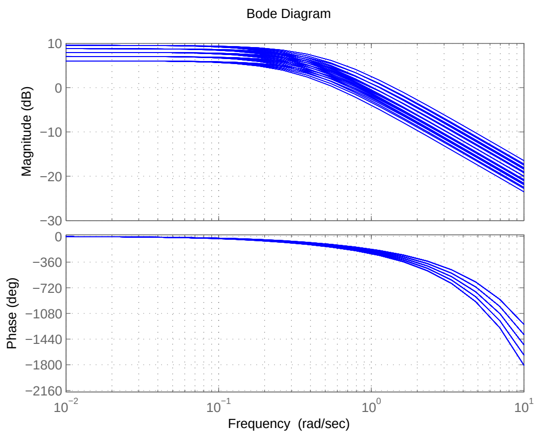
Matlab:

- **lft** (Control Systems Tbx), **lftdata** (Robust Control Tbx)
- LFR Toolbox for Matlab (J.F.Magni, last update 2014)

Real parametric uncertainty cast as dynamic

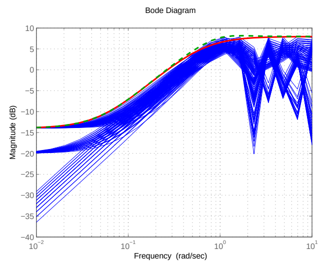
Consider an uncertain system

$$G(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \qquad 2 \le k, \tau, \theta, \le 3$$



Real parametric uncertainty cast as dynamic—contd.

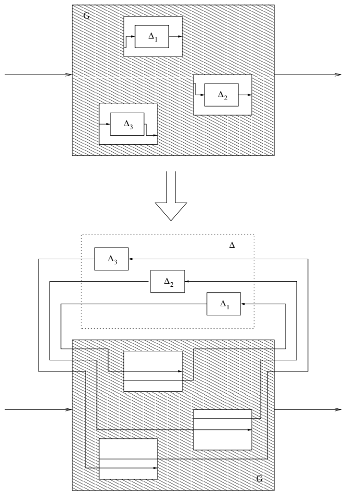
$$\frac{|G(j\omega) - G_{nom}(j\omega)|}{|G_{nom}(j\omega)|} < |W(j\omega)|$$



$$\boxed{W(s) = \frac{\tau s + r_0}{(\tau / r_\infty)s + 1}}$$

where r_0 is uncertainty at steady state, $1/\tau$ is the frequency, where the relative uncertainty reaches 100%, r_∞ is relative uncertainty at high frequency, often $r_\infty \ge 2$.

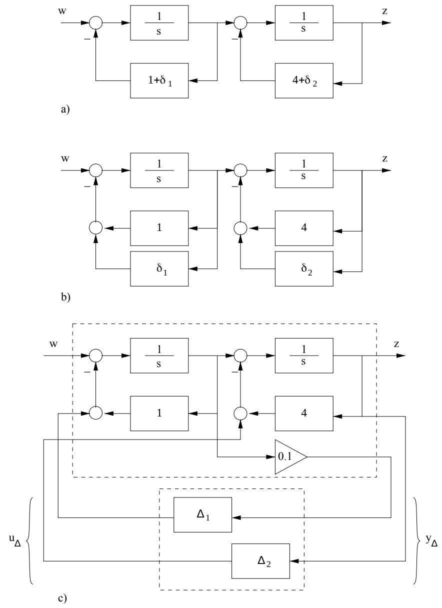
Structured dynamic uncertainty = more Δ blocks



Example of a structured dynamic uncertainty

$$G(s) = \frac{1}{(s+p_1)(s+p_2)}$$
$$p_1 \in [0.9, 1], p_2 \in [3, 5]$$

$$N(s) = \begin{bmatrix} N_{y_\Delta u_\Delta}(s) & N_{y_\Delta w}(s) \\ N_{z u_\Delta}(s) & N_{zw}(s) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{-0.1}{s+1} & 0 & \frac{0.1}{s+1} \\ \frac{-1}{(s+1)(s+4)} & \frac{-1}{s+4} & \frac{1}{(s+1)(s+4)} \\ \frac{-1}{(s+1)(s+4)} & \frac{-1}{s+4} & \frac{1}{(s+1)(s+4)} \end{bmatrix}$$



Entering dynamic uncertainty models in Matlab

- Robust Control Tbx: **ultidyn** class
- LFRT toolbox (J.-F. Magni)

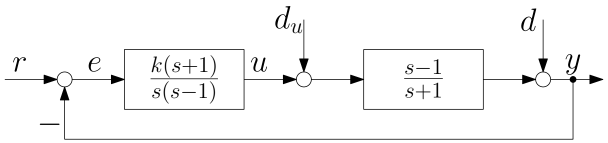
Two central properties for which robustness matters

Robust stability is guaranteed stability of the closed feedback loop with a given controller for all admissible (=considered apriori) deviations of the model from the reality.

Robust performance is robustness of some performance characteristics such as steady-state regulation error, attenuation of some specified disturbance, insensitivity to measurement noise, fast response, ...).

Internal stability

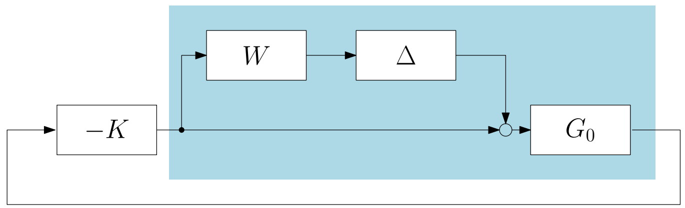
All the signals in the loop must be bounded for **all** possible inputs
$\Longleftrightarrow$ all the possible closed-loop transfer functions must be stable



$$T = \frac{GK}{1+GK} = \frac{\frac{k}{s}}{1+\frac{k}{s}} = \frac{k}{s+k} \qquad S = \frac{1}{1+GK} = \frac{1}{1+\frac{k}{s}} = \frac{s}{s+k}$$

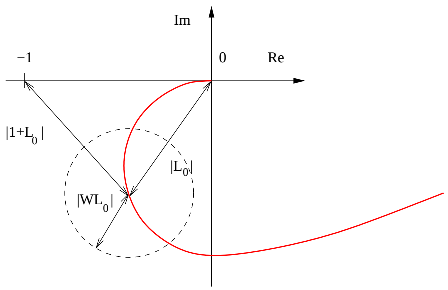
$$U = \frac{\hat{u}(s)}{\hat{d}(s)} = \frac{-K}{1+GK} = \frac{-\frac{k(s+1)}{s(s-1)}}{1+\frac{k}{s}} = \frac{-k(s+1)}{(s-1)(s+k)}$$

Robust stability (RS) for multiplicative uncertainty



$$L = GK = G_0K(1+W\Delta) = L_0 + L_0W\Delta, \quad |\Delta(j\omega)| \le 1, \forall\omega$$

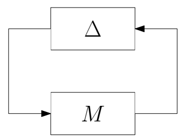
Robust stability (RS) for multiplicative uncertainty using Nyquist criterion



$$\begin{aligned} \text{RS} &\Leftrightarrow |W(j\omega)L_0(j\omega)| < |1+L_0(j\omega)|, \ \forall\omega \\ &\Leftrightarrow \left|\frac{W(j\omega)L_0(j\omega)}{1+L_0(j\omega)}\right| < 1, \ \forall\omega \\ &\Leftrightarrow |T(j\omega)| < 1/|W(j\omega)|, \ \forall\omega \\ &\Leftrightarrow \|WT\|_\infty \le 1 \end{aligned}$$

Robust stability for LFT—Small Gain Theorem

$$\mathcal{F}_u(\mathrm{N},\Delta) = N_{22} + N_{21}\Delta(I - \underbrace{N_{11}}_{M}\Delta)^{-1}N_{12}$$



$$|1 - M(j\omega)\Delta(j\omega)| > 0, \ \forall\omega, \forall|\Delta| \le 1$$

$$\begin{aligned} RS &\Leftrightarrow 1 - |M(j\omega)| > 0, \ \forall\omega \\ &\Leftrightarrow |M(j\omega)| < 1, \ \forall\omega \\ &\Leftrightarrow \|M\|_\infty < 1 \end{aligned}$$

Nominal performance (NP)

Using **sensitivity function** (for MIMO systems it is necessary to distinguish between *input* a *output* sensitivities)

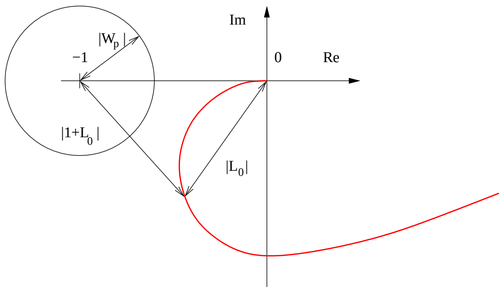
$$S_0(s) = \frac{1}{1 + G_0(s)K(s)}$$

can express

- requirements on bandwidth (S is typically high-pass),
- largest acceptable tracking error,
- type of system (number of integrators),
- damping (bound on resonance peak)

Conditions for nominal performance (NP)

$$|S_0(j\omega)| < 1/|W_p(j\omega)|, \;\; \forall\omega \qquad \|W_p S_0\|_\infty < 1$$



$$|W_p(j\omega)| < |1 + L_0(j\omega)| \;\; \forall\omega$$

Simple "templates" usually suffice

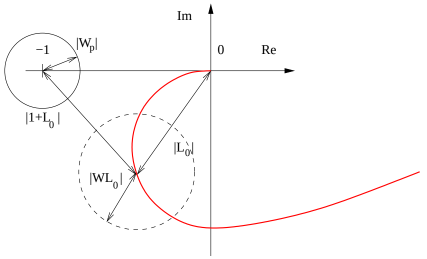
$$W_p(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A} \qquad W_p(s) = \frac{(s/\sqrt{M} + \omega_B^*)^2}{(s + \omega_B^*\sqrt{A})^2}$$

Robust performance (RP) for multiplicative uncertainty: $RP \neq RS + NP$

$$\begin{aligned} RP &\Leftrightarrow |W_p(j\omega)S(j\omega)| < 1 \quad \forall S, \forall\omega \\ &\Leftrightarrow |W_p(j\omega)| < |1 + L(j\omega)| \quad \forall L, \forall\omega \end{aligned}$$

$$L = GK = G_0K(1 + W\Delta) = L_0 + WL_0\Delta, \quad |\Delta(j\omega)| \leq 1, \forall\omega$$

$$RP \Leftrightarrow |W_p(j\omega)| < |1 + L_0(j\omega) + W(j\omega)L_0(j\omega)\Delta(j\omega)| \quad \forall\Delta, \forall\omega$$



Condition for RP for multiplicative uncertainty

$$|1 + L_0(j\omega)| > |W_p(j\omega)| + |W(j\omega)L_0(j\omega)|, \;\; \forall\omega$$

$$\boxed{|W_p(j\omega)S(j\omega)| + |W(j\omega)T(j\omega)| < 1 \;\; \forall\omega}$$

Not in the format of a condition on the $\mathcal{H}_\infty$ system norm. But recall that for SISO systems

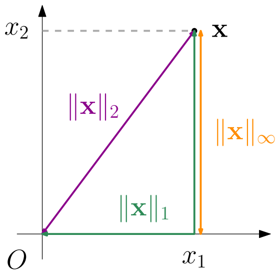
$$\left\| \underbrace{\begin{bmatrix} W_pS \\ WT \end{bmatrix}}_{\text{mixed sensitivity}} \right\|_\infty = \sup_{\omega\in\mathbb{R}} \sqrt{|W_p(j\omega)S(j\omega)|^2 + |W(j\omega)T(j\omega)|^2}$$

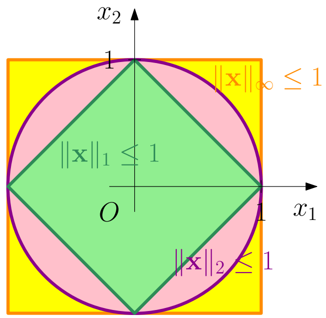
Hint: $\sigma_i(\mathrm{A}) = \sqrt{\lambda_i(\mathrm{A}^\mathsf{T}\mathrm{A})}$

Condition for RP for multiplicative uncertainty

Recall various norms in the Euclidean plane

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}, \quad \|x\|_1 = |x_1| + |x_2|, \quad \|x\|_\infty = \max\{|x_1|, |x_2|\}$$





$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{2}\|x\|_2$$

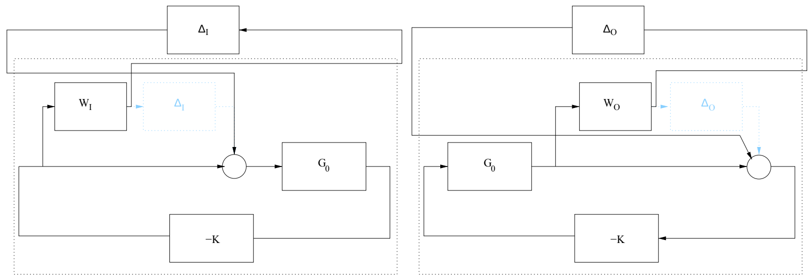
Therefore RP can be formulated (conservatively) as

$$\left\| \begin{bmatrix} W_p S \\ WT \end{bmatrix} \right\|_\infty < \frac{1}{\sqrt{2}}$$

Conditions of robust stability for MIMO systems

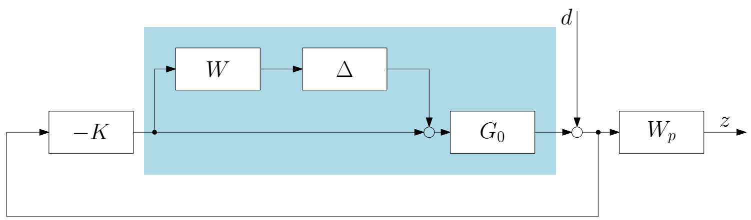
Generally the model of uncertainty is $E = W_2 \Delta W_1, \quad \|\Delta\|_\infty \leq 1$.
Necessary to distinguish uncertainty at the **input** and **output**. For multiplicative uncertainty

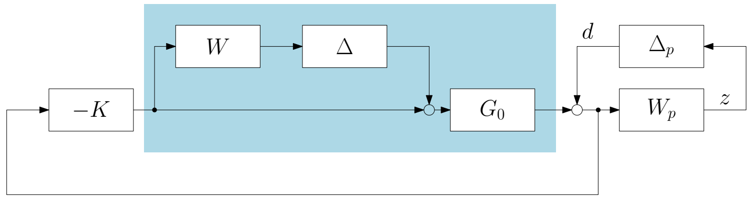
$$G = G_0(I + E_I), \qquad G = (I + E_O)G_0$$



$$M_I = -W_I K G_0 (I + K G_0)^{-1} = -W_I T_I$$
$$M_O = -W_O G_0 K (I + G_0 K)^{-1} = -W_O T_O$$

Robust performance as robust stability with structured uncertainty





Structured dynamic uncertainty: μ-analysis

In $M\Delta$ configuration, with Δ having some structure

$$\Delta = \begin{bmatrix} \Delta_1 & 0 & \dots & 0 \\ 0 & \Delta_2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \Delta_n \end{bmatrix}$$

$$\|\Delta_i\|_\infty \leq 1, \; i = 1, \dots, n$$

Conservativeness of the small gain theorem in MIMO case

$$\sup_\omega \bar\sigma(\mathrm{M}(j\omega)) < \frac{1}{\sup_\omega \bar\sigma(\Delta(j\omega))} \quad \Leftrightarrow \quad \sup_\omega \bar\sigma(\Delta(j\omega)) < \frac{1}{\sup_\omega \bar\sigma(\mathrm{M}(j\omega))}$$

Example:

$$\mathrm{M} = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

The smallest destabilizing perturbation (when $\det(\mathrm{I} - \mathrm{M}\Delta) = 0$) is

$$\Delta = \frac{1}{\bar\sigma(\mathrm{M})} \mathrm{V}_1 \mathrm{U}_1^*, \qquad \mathrm{M} = \mathrm{U}\Sigma\mathrm{V}^*, \qquad 1/\bar\sigma_1(\mathrm{M}) = 0.3162$$

However, if we only consider diagonal uncertainty

$$\det\left(\mathrm{I} - \mathrm{M}\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\right) = 0 \Rightarrow d_1 = d_2/2 + 1/2$$

Solution minimizing $\max\{|d_1|, |d_2|\}$, is $d_1 = 1/3, d_2 = -1/3 \Rightarrow$

$$\Delta = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

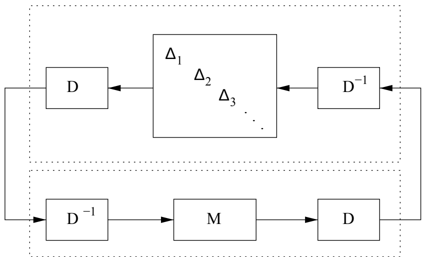
Structured singular value—definition

$$\mu_\Delta(\mathrm{M}) \triangleq \frac{1}{\min_\Delta \{\bar\sigma(\Delta) |\det(\mathrm{I} - \mathrm{M}\Delta) = 0 \text{ given the structure of } \Delta\}}$$

$$\mu_\Delta(\mathrm{M}) \triangleq \frac{1}{\min_{\Delta, k_M} \{k_M| \det(\mathrm{I} - k_M \mathrm{M}\Delta) = 0 \text{ for struct. } \Delta, \bar\sigma(\Delta) \leq 1\}}$$

Methods to calculate SSV: upper bound

For full (unstructured) uncertainty, the largest freedom for uncertainty $\mu(\mathrm{M}) = \bar\sigma(\mathrm{M})$. Too conservative.
To decrease conservativeness: **D-scaling**



$$\mathrm{D} = \begin{bmatrix} d_1 \mathrm{I}_1 & 0 & \dots & 0 \\ 0 & d_2 \mathrm{I}_2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & d_n \mathrm{I}_n \end{bmatrix}$$

$$\Delta = \Delta \mathrm{D}\mathrm{D}^{-1} = \mathrm{D}\Delta\mathrm{D}^{-1}$$

Methods to calculate SSV: upper bound

New (tighter) condition of robust stability

$$\|\mathrm{D}\mathrm{M}\mathrm{D}^{-1}\|_\infty < 1, \qquad \mathrm{D}(\omega) \in \mathcal{D}$$

$$\min_{\mathrm{D}(\omega) \in \mathcal{D}} \bar\sigma(\mathrm{D}(\omega)\mathrm{M}(j\omega)\mathrm{D}^{-1}(\omega)) < 1, \ \forall\omega$$

Convex (linear matrix inequality, LMI). Up to 3 blocks exactly, beyond 3 blocks error about 15%.
Robust Control Toolbox: **mussv**

Example: structured (diagonal) uncertainty at the input

Simplified model of a distillation column in DV configuration

$$\mathrm{G}(s) = \frac{1}{75s+1}\begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix}$$

and decentralized (diagonal matrix of transfer fcn's) PI regulator

$$\mathrm{K}(s) = \frac{75s+1}{s}\begin{bmatrix} -0.0015 & 0 \\ 0 & -0.075 \end{bmatrix}$$

Uncertainty modeled as multiplicative at *every* input: about 20 % at low frequency, and 100 % above 1 rad/s:

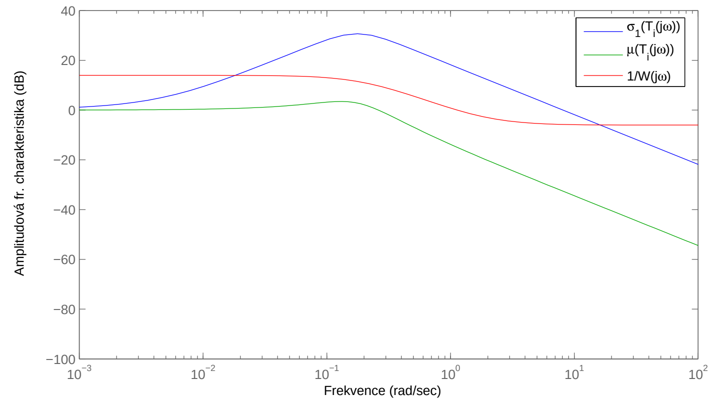
$$\mathrm{W_I}(s) = \begin{bmatrix} w_\mathrm{I}(s) & 0 \\ 0 & w_\mathrm{I}(s) \end{bmatrix}; \quad w_\mathrm{I}(s) = \frac{s+0.2}{0.5s+1}$$

the block Δ is structured

$$\Delta(s) = \begin{bmatrix} \Delta_1(s) & 0 \\ 0 & \Delta_2(s) \end{bmatrix}$$

Is the system robustly stable?

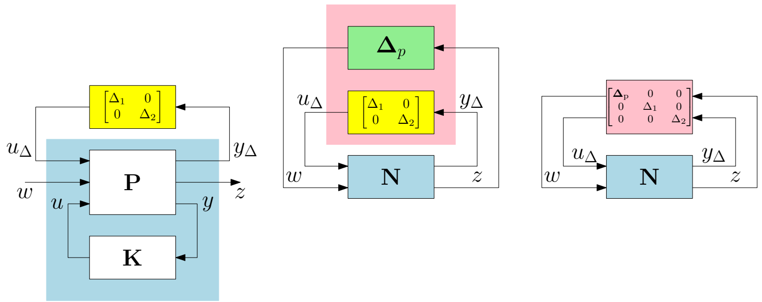
Example: structured (diagonal) uncertainty at the input



$$\mu_\Delta(\mathrm{T}(j\omega)) < \frac{1}{w_\mathrm{I}(j\omega)}, \ \forall\omega, \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

Robust performance with structured uncertainty

The total structured uncertainty includes also a ficticious (full, unstructured) uncertainty Δ_P



Example: robust performance of distillation column

Linear model of distillation column in LV configuration (flows are the action inputs, concentrations are the outputs)

$$\mathrm{G}(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$$

and some designed controler

$$\mathrm{K}(s) = \frac{0.7}{s}\mathrm{G}(s)^{-1}$$

$$\mathrm{L}(s) = \begin{bmatrix} \frac{0.7}{s} & 0 \\ 0 & \frac{0.7}{s} \end{bmatrix}$$

Sensitivity and complementary sensitivity functions are

$$\mathrm{S}(s) = \begin{bmatrix} \frac{s}{s+0.7} & 0 \\ 0 & \frac{s}{s+0.7} \end{bmatrix}, \quad \mathrm{T}(s) = \begin{bmatrix} \frac{0.7}{s+0.7} & 0 \\ 0 & \frac{0.7}{s+0.7} \end{bmatrix}$$

Example: robust performance of a distillation column

The inputs are flows and the valves have always (at least 10%) $\Rightarrow$ multiplicative uncertainty at **every** input.

$$w_I(s) = \frac{s+0.2}{0.5s+1}$$

The requirements on performance

$$w_P(s) = \frac{s/2+0.05}{s}$$

Four major properties of the closed loop need to be checked

- nominal stability
- nominal performance
- robust stability
- robust performance

Example: robust performance of distillation column

NS: no pole-zero cancellation, suffices to check, say, S
NP: $\Delta = 0 \Rightarrow \mathcal{F}_u(N,\Delta) = N_{22}$, where

$$N(s) = \begin{bmatrix} w_I T_I & w_I KS \\ w_P SG & w_P S \end{bmatrix}$$

with the given controller we get

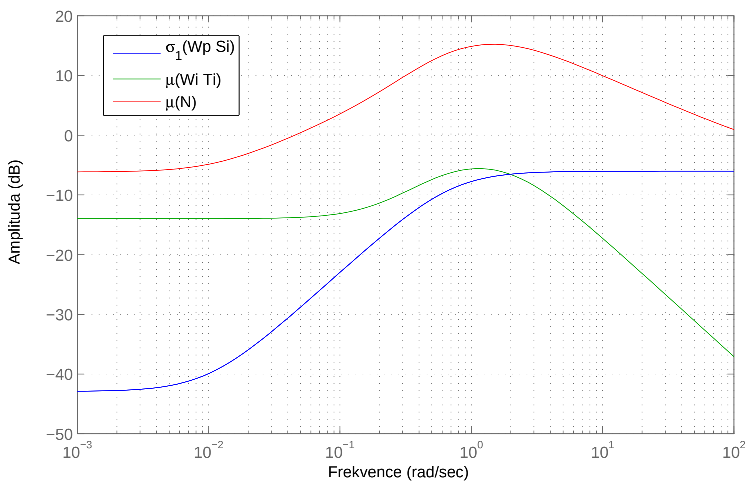
$$\bar{\sigma}(N_{22}) = \bar{\sigma}(w_P S) = \left| \frac{s/2+0.05}{s+0.7} \right|$$

RS: only the block N_{11}:

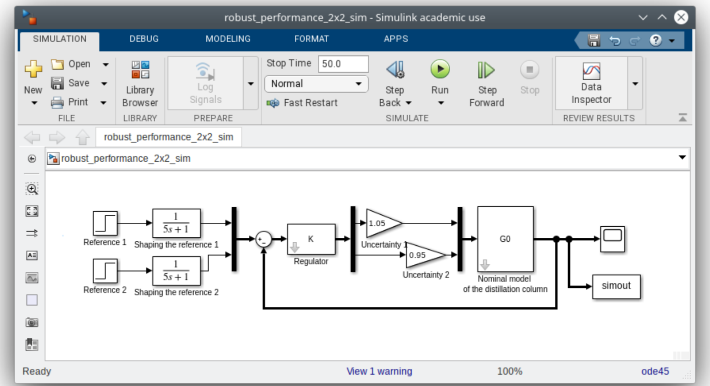
$$\mu_\Delta(N_{11}) = \mu_\Delta(w_I T_I) = 0.2\frac{5s+1}{(0.5s+1)(1.43s+1)}$$

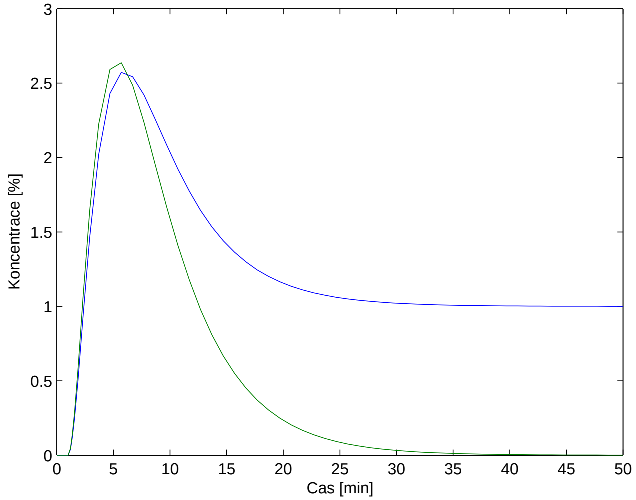
RP:full N for μ analysis

Example: robust performance of distillation column



Example: robust performance of distillation column




3
2.5
2
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1
0.5
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0
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10
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Koncentrace [%]
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