Design of robust controller by minimization of $\mathcal{H}_\infty$ system norm

Graduate course on Optimal and Robust Control

Zdeněk Hurák

Department of Control Engineering
Faculty of Electrical Engineering
Czech Technical University in Prague

April 28, 2021

Lecture outline

Mixed sensitivity minimization

General $\mathcal{H}_\infty$ norm minimization problem

μ-synthesis via DK iterations

Classical vs. modern methods

Classical methods require that you know what you want and how to achieve it.

Modern methods require that you only know what you want. Don't bother with how to achieve it.

How to express what you want

The already known condition of robust performance under multiplicative uncertainty

$$\left\| \begin{bmatrix} W_p S \\ W T \end{bmatrix} \right\|_\infty < 1$$

Turn into optimization task (and solve using **mixsyn()**)

$$\min_{K \text{ stabilizing}} \left\| \begin{bmatrix} W_p S \\ W T \end{bmatrix} \right\|_\infty$$

Or more generally

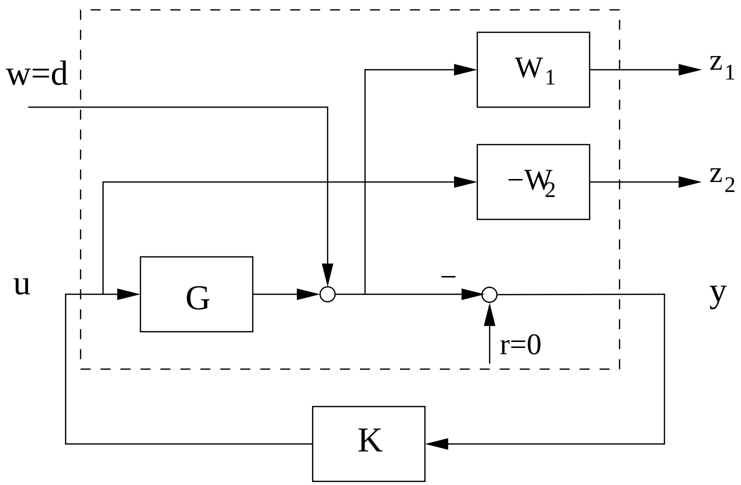
$$\min_{K \text{ stabilizing}} \left\| \begin{bmatrix} W_1 S \\ W_2 K S \\ W_3 T \end{bmatrix} \right\|_\infty$$

The middle term penalizes control (similarly as R term in LQ optimality criterion $\int (x^T Q x + u^T R u) dt$).

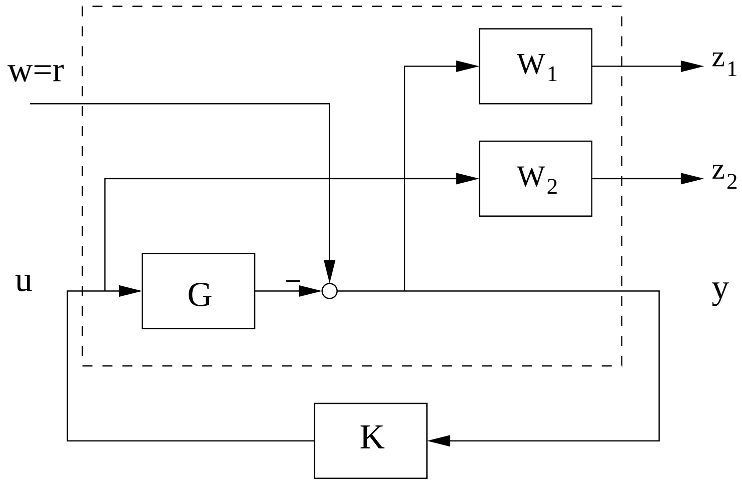
Two-term S/KS mixed sensitivity problem

When no tracking is needed

$$\min_{K \text{ stabilizing}} \left\| \begin{bmatrix} W_1 S \\ W_2 KS \end{bmatrix} \right\|_\infty$$



Another intepretation of S/KS mixed sensitivity minimization - tracking

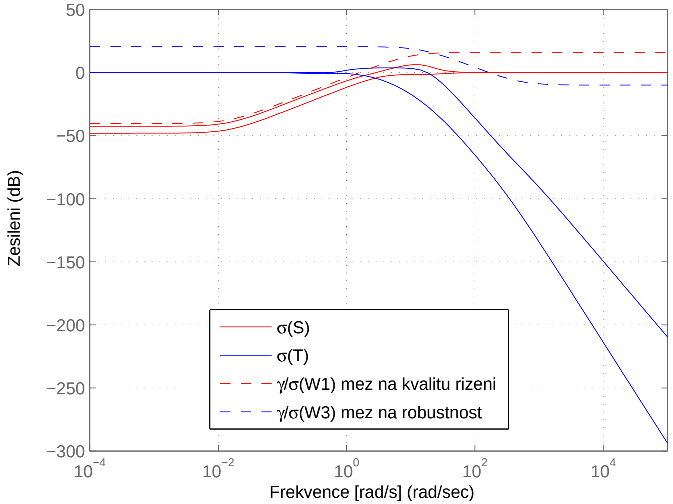


Two-term S/T mixed sensitivity problem

Constraint on Ws

Filters must be stable and proper. $W_p(s) = \frac{1}{s}$ should be modified to $W_p(s) = \frac{1}{s+\epsilon}$ and $W_u = 1 + \tau_u s$ should be modified to $W_u = \frac{1+\tau_u s}{1+\frac{\tau_u}{\alpha}s}$, $\alpha >> 1$.

Ex: Mixed sensitivity (himat)



Formulation of control design problem using signal interpretation of $\mathcal{H}_\infty$ system norm

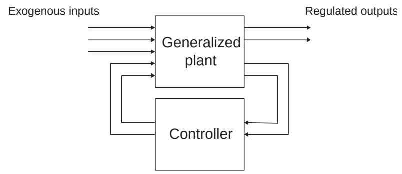
Recall

$$\|G\|_\infty = \sup_{u \in \mathcal{L}_2 \setminus \emptyset} \frac{\|y\|_2}{\|u\|_2}$$

Formulate the generalized plant P such that it makes sense to minimize

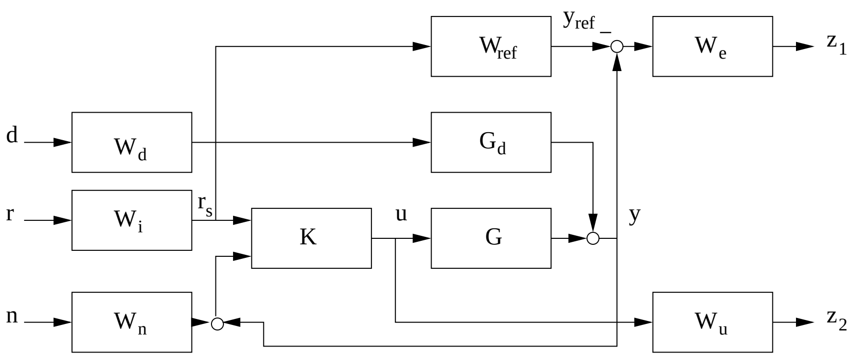
$$\min_{K \text{ stabilizing}} \|F_l(P,K)\|_\infty$$

by searching among stabilizing controllers K.



Solve using **hinfsyn()** in Matlab. Also functions in Scilab, Octave, Slicot, Mathematica, Maple, ...

Typical desig problem solved by general $\mathcal{H}_\infty$ system norm minimization



Mixed S/KS sensitivity cast as general $\mathcal{H}_\infty$ problem

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

For disturbance rejection

$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix}; \quad P_{12} = \begin{bmatrix} W_1 G \\ -W_2 \end{bmatrix}; \quad P_{21} = -I; \quad P_{22} = -G$$

For tracking

$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix}; \quad P_{12} = \begin{bmatrix} -W_1 G \\ W_2 \end{bmatrix}; \quad P_{21} = I; \quad P_{22} = -G$$

Conditions of existence of $\mathcal{H}_\infty$ optimal controller

$$P(s) = \left[\begin{array}{c|cc} A_1 & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Conditions for existence of K minimizing $\|F_l(P,K)\|_\infty$

1. system (A, B_2, C_2) is stabilizable and detectable,
2. matrices D_{12} and D_{21} are full rank, otherwise the controller has high gain at high frequency (nonproper),
3. matrix $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ full column rank for all ω,
4. matrix $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ full row rank for all ω (to prevent low damped poles of the controller),
5. $D_{11} = 0$ and $D_{22} = 0$. Not necessary. Simplifies solution.

Algorithm for solving the $\mathcal{H}_\infty$ optimization

Search for a positive definite $X_\infty \geq 0$ and $Y_\infty \geq 0$ solving

$$\begin{aligned} A^T X_\infty + X_\infty A + C_1^T C_1 + X_\infty(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty &= 0 \\ A Y_\infty + Y_\infty A^T + B_1 B_1^T + Y_\infty(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y_\infty &= 0 \end{aligned}$$

such that $\Re\lambda_i(A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty) < 0, \ \forall i$ and $\Re\lambda_i(A + Y_\infty(\gamma^{-2} C_1^T C_1 - C_2^T C_2)) < 0, \ \forall i$ and satisfying

$$\rho(X_\infty Y_\infty) < \gamma^2$$

All stabilizing controllers given by LFT $K = F_l(K_c, Q)$, where K_c is

$$K_c(s) = \left[\begin{array}{c|cc} A_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

$$F_\infty = -B_2^T X_\infty, L_\infty = -Y_\infty C_2^T, Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}$$
$$A_\infty = A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$

and $Q(s)$ is an arbitrary stable transfer function satisfying $\|Q\|_\infty < \gamma$. For $Q = 0$

$$K(s) = K_{c11}(s) = -Z_\infty L_\infty (sI - A_\infty)^{-1} F_\infty$$

Central regulator, same order as P.

Structure of $\mathcal{H}_\infty$ optimal controller

Central controller can be, similarly as LQG controller, separated into estimator

$$\dot{\hat{x}} = A\hat{x} + B_1^T \underbrace{\gamma^{-2} B_1^T X_\infty \hat{x}}_{\hat{w}_{\text{worst}}} + B_2 u + Z_\infty L_\infty (c_2 \hat{x} - y)$$

and state feedback

$$u = F_\infty \hat{x}$$

Compared to Kalman filter: extra term $B_1 \hat{w}_{\text{worst}}$, where $\hat{w}_{\text{worst}}$ can be interpretted as an estimate of worst case disturbance.

Design for robust stability with structured uncertainty

Condition of robust stability

$$sup_\omega \mu_{\hat{\Delta}}(N(j\omega)) = \sup_\omega \mu_{\hat{\Delta}}(F_l(P(j\omega), K(j\omega))) < 1$$

where

$$\hat{\Delta}(s) = \begin{bmatrix} \Delta_1(s) & & & & 0 \\ & \Delta_2(s) & & & \\ & & \ddots & & \\ & & & \Delta_k(s) & \\ 0 & & & & \Delta_P(s) \end{bmatrix}$$

Control design formulated as

$$\min_{K \text{ stabilizing}} \sup_\omega \mu_{\hat{\Delta}}(F_l(P(j\omega), K(j\omega)))$$

D-scaling and interpolation

Related (feasible) optimization problem using upper bound on μ

$$\underbrace{\min_{K \text{ stabilizing}}}_{\text{K step}} \underbrace{\min_{D \text{ stable and min. phase}}}_{\text{D step}} \|DND^{-1}\|_\infty$$

Outcome of D-scaling: constant matrices D_i, $i = 1, 2, \dots, 1000$. Interpolation to obtain stable and minimum phase $D(s)$.

Theorem (Bode's relationship between magnitude and phase responses)

For scalar stable and minimum phase G normalized such that $G(0) > 0$

$$\angle G(j\omega_l) = \frac{2\omega_l}{\pi} \int_0^\infty \frac{\ln|G(j\omega)| - \ln|G(j\omega_l)|}{\omega^2 - \omega_l^2} d\omega$$

In Matlab: **fitmagfrd()**

DK iterations

1. Set initial values of the scaling filters in $D(s) = \text{diag}(d_1(s), d_2(s), \dots, d_k(s), d_P(s)I)$ to I
2. Scale the system $\text{diag}(D, I)P\text{diag}(D, I)^{-1}$ and find K minimizing $\|DND^{-1}\|_\infty$, where $N = F_l(P, K)$
3. Find D_i, $i = 1, 2, \dots, k$ matrices for a set of k frequencies ω_i
4. Interpolate D_i, $i = 1, 2, \dots, k$ with stable and minimum phase $D(s)$
5. If $\mu < 1$ or if it did not change since the last iteration step, finish, otherwise go to step 2.

In Matlab: **dksyn()**

Critical points in DK iterations

- Nonconvex task, convergence not guaranteed
- Order of the interpolating filters $D(s)$ adds to the order of the controller
- μ and $\mathcal{H}_\infty$ methodologies focus on worst case, the more deltas the more conservative design