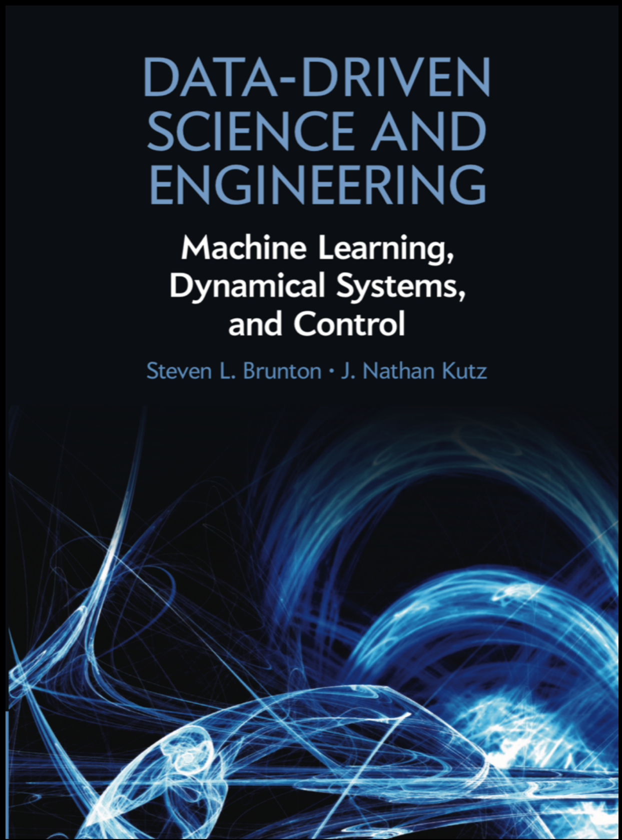
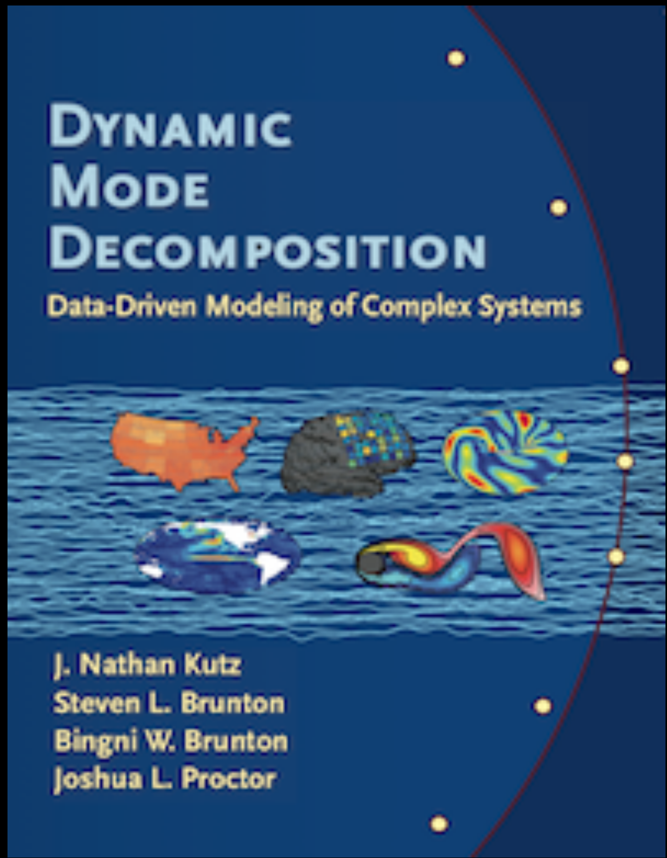
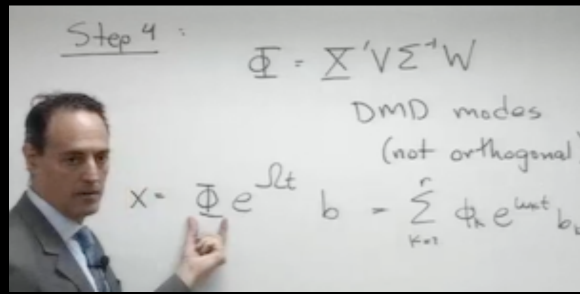
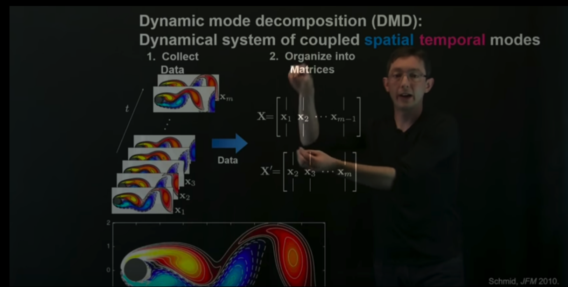
Dynamic mode decomposition (DMD) & Koopman operator theory – „meta lecture"

Accesible introduction in books and videos by Steve Brunton and Nathan Kutz









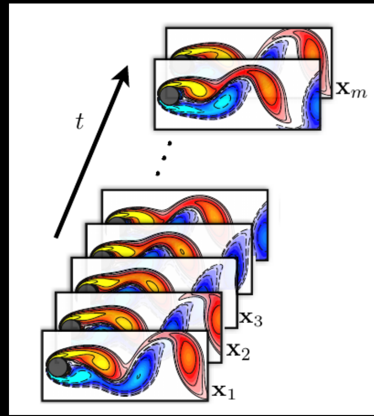
Dynamic mode decomposition (DMD)

Given: a sequence of „snapshots"

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & \dots & x_m \end{array}$$
$$n\left\{\begin{array}{ccccc} \vert & \vert & \vert & \dots & \vert \\ \vert & \vert & \vert & & \vert \end{array}\right.$$

Can come from
- experiments
- simulations

Very high dimension n



[Brunton (2019)]

Goal: Find a LINEAR model $\boxed{x_{k+1} \approx A\, x_k}$
of REDUCED order

$$\underset{A \in \mathbb{R}^{n \times n}}{\text{minimize}} \sum_{k=1}^{m-1} \| A\, x_k - x_{k+1} \|_2$$

$$\Downarrow$$

$$X = \begin{bmatrix} \vert & \vert & & \vert \\ x_1 & x_2 & \cdots & x_{m-1} \\ \vert & \vert & & \vert \end{bmatrix}$$

$$X' = \begin{bmatrix} \vert & \vert & & \vert \\ x_2 & x_3 & \cdots & x_m \\ \vert & \vert & & \vert \end{bmatrix}$$

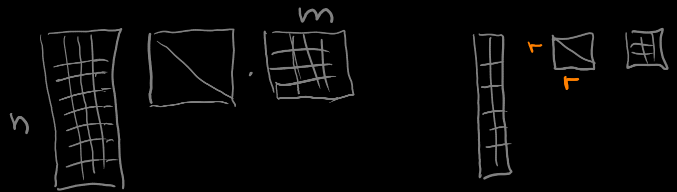
$$\boxed{A = \arg\min_A \| AX - X' \|_F}$$

Solution using pseudo-inverse

$$A = \arg\min_{A} \|AX - X\|_F = X'X^{+}$$

... can be computed using SVD

$$X = U \cdot \Sigma \cdot V^* = \tilde{U}\tilde{\Sigma}\tilde{V}^*$$



$$A = X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^* \xrightarrow{\text{proper orthogonal decomp.}} \boxed{\tilde{A} = \tilde{U}^* A \tilde{U} = \tilde{U}^* X' \tilde{V}\tilde{\Sigma}^{-1}}$$

$$\Downarrow$$

$$\boxed{\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k}$$

reduced order

$\underbrace{\text{Eigendecomposition}}_{\text{also modal}}$ of $\tilde{A}$: $\boxed{\tilde{A}W = W\Lambda}$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$$

Project the modes into the original high dimension

$$\cancel{\phi_i = \tilde{U} \cdot w_i} \qquad \phi_i = X'\tilde{V}\tilde{\Sigma}^{-1} w_i \qquad \boxed{\underbrace{\Phi}_{\begin{bmatrix} | & | & & | \\ \phi_1 & \phi_2 & \cdots & \phi_r \\ | & | & & | \end{bmatrix}} = X'\tilde{V}\tilde{\Sigma}^{-1} W}$$

Why compute the modes ϕ_i ?

$$\boxed{x_k = \sum_{i=1}^{r} \phi_i \lambda_i^{k-1} b_i = \Phi \Lambda^{k-1} b}$$

$$\Phi b = x_1$$

$$b = \Phi^{\dagger} x_1$$

Can also convert to continuous time

$$\boxed{x(t) = \sum_{i=1}^{r} \phi_i e^{\omega_i t} b_i = \Phi e^{\Omega t} b}$$

CONT.	DISCRETE
p $\longrightarrow$	e^{pT}
$\underbrace{\frac{\ln \lambda}{T}}_{\omega}$ $\longleftarrow$	λ

DMD = „kind of blend" of PCA/POD with Fourier

Give it a try with data from dmdbook.com

Obviously some nonlinear phenomena can't be captured by $x_{k+1} = A x_k$

- multiple equilibria (pendulum)
- limit cycles
- ...

$\Downarrow$

EXTENDED DMD (eDMD)

observables (also features)

$$\begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_p(m) \end{bmatrix}$$

typically $p \gg n$

$$\left.\begin{aligned} Y &= \begin{bmatrix} | & | & & | \\ g(x_1) & g(x_2) & \cdots & g(x_{m-1}) \\ | & | & & | \end{bmatrix} \\ Y' &= \begin{bmatrix} | & | & & | \\ g(x_2) & g(x_3) & \cdots & g(x_m) \\ | & | & & | \end{bmatrix} \end{aligned}\right\}$$

$$A_Y = \underset{A_Y}{\arg\min} \|Y' - A_Y Y\|_F = Y' Y^+$$

Easy transformation back to x if <u>some</u> $g_i(x) = x_1$

$g_{i+1}(x) = x_2$

$\vdots$

Koopman operator

Nonlinear state-space model $\dot{x}(t) = \underset{\text{nonlinear}}{f}(x(t))$, $x \in \mathbb{R}^n$ finite-dimensional

Introduce „observable"

$$y(t) = g(x(t)) = \begin{bmatrix} g_1(x(t)) \\ g_2(x(t)) \\ \vdots \\ g_p(x(t)) \end{bmatrix} \quad \text{but can be just} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Given $x(0)$, we get $\boxed{x(t) = \mathcal{F}_t(x(0))}$ LTI system: $\boxed{\mathcal{F}_t = e^{At}}$

How get $y(t)$?

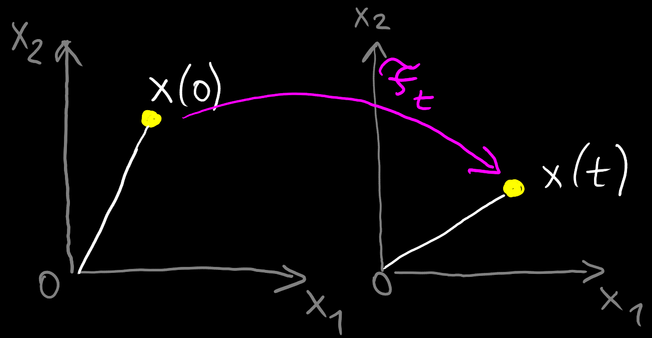
$$y(t) = \underbrace{g(\mathcal{F}_t}_{\text{composition}}(x(0))) = (\mathcal{K}_t g)(x(0)) \qquad \boxed{\mathcal{K}_t g = g \circ \mathcal{F}_t}$$

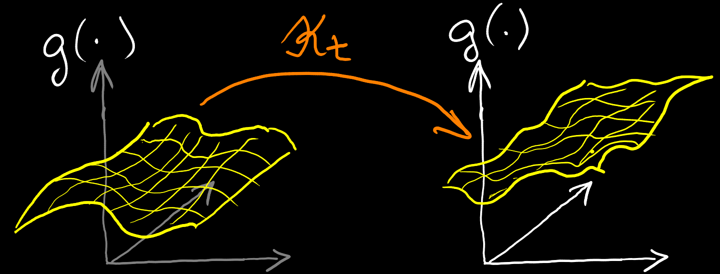
KOOPMAN OPERATOR $\mathcal{K}_t$ (acts on functions)

$\Downarrow$

infinite-dimensional
but
linear

$$\mathcal{K}_t(\alpha_1 g_1(x) + \alpha_2 g_2(x)) = \alpha_1 \mathcal{K}_t g_1(x) + \alpha_2 \mathcal{K}_t g_2(x)$$





For discretized (uniformly sampled) systems

$$y_{k+1} = g(x_{k+1}) = g(\mathcal{F}_{\Delta t}(x_k)) = (\mathcal{K}_{\Delta t}\, g)(x_k)$$

Example: LTI discrete-time system

$$x_{k+1} = A x_k, \quad x_0 \text{ given} \implies \mathcal{F}_1 = A$$

$$y_k = C x_k \implies g(x_k) = C x_k$$

$$y_{k+1} = CA\, x_k \implies \mathcal{K}_{\Delta t} : C \longrightarrow CA$$

$$y_k = CA^k x_0 \implies \mathcal{K}_{k \cdot \Delta t} : C \longrightarrow CA^k$$

Example:

$$\dot{x}_1 = \mu x_1$$
$$\dot{x}_2 = \lambda \cdot (x_2 - x_1^2)$$

Observables:

$$y_1 = x_1 \qquad \dot{y}_1 = \dot{x}_1 = \mu x_1 = \mu x_1$$
$$y_2 = x_2 \qquad \dot{y}_2 = \dot{x}_2 = \lambda \cdot (x_2 - x_1^2) = \lambda \cdot (y_2 - y_3)$$
$$y_3 = x_1^2 \qquad \dot{y}_3 = \frac{d}{dt} x_1^2 = 2 x_1 \cdot \dot{x}_1 = 2\mu x_1^2 = 2\mu y_3$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & 2\mu \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Questions: 1) when finite-dimensional Koopman model ?

2) how to find suitable observables ?

Koopman LQ-optimal control

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\mu & 0\\ 0 & \lambda\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\underbrace{\begin{bmatrix}0\\ -\lambda\cdot x_1^2\end{bmatrix}}_{\approx 0}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$\min_{u}\int_0^\infty\left[\,[x_1\ \ x_2]\cdot\underbrace{\begin{bmatrix}q_1 & 0\\ 0 & q_1\end{bmatrix}}_{Q}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+R\cdot u^2\right]dt$$

$$\boxed{u=-[k_1\ \ k_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix}}$$

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix}=\begin{bmatrix}\mu & 0 & 0\\ 0 & \lambda & -\lambda\\ 0 & 0 & 2\mu\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}+\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}u$$

$$\min_{u}\int_0^\infty\left[\,[y_1\ \ y_2\ \ y_3]\cdot\underbrace{\begin{bmatrix}q_1 & & \\ & q_2 & \\ & & 0\end{bmatrix}}_{\tilde{Q}}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}+R u^2\right]dt$$

$$\boxed{u=-[\tilde{k}_1\ \ \tilde{k}_2]\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}-\tilde{k}_3\cdot x_1^2}$$

Koopman model in general ∞-dimensional

Example: $\dot{x} = x^2$

$$y_1 = x \qquad \dot{y}_1 = \dot{x} = x^2 = y_2$$

$$y_2 = x^2 \qquad \dot{y}_2 = \frac{d}{dt} x^2 = 2x \cdot \dot{x} = 2 y_1 \cdot y_1^2 = 2 y_1^3 = 2x^3$$

$$y_3 = x^3 \qquad \dot{y}_3 = \frac{d}{dt} x^3 = 3x^2 \cdot \dot{x} = 3 \cdot y_1^2 \cdot y_1^2 = 3 \cdot y_1^4 = 3x^4$$

$$y_4 = x^4$$

$$y_5 = x^5$$

$$\vdots$$

Truly infinite-dimensional $\Rightarrow$ TRUNCATING:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \vdots \\ \dot{y}_8 \\ \dot{y}_9 \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 2 & & & & \\ & & 0 & 3 & & & \\ & & & 0 & 4 & & \\ & & & & \ddots & \ddots & \\ & & & & & 0 & 8 \\ & & & & & & 0 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_8 \\ y_9 \end{bmatrix}$$

NEVER finite-dimensional if

multiple equilibria, or periodic orbit, ...

&

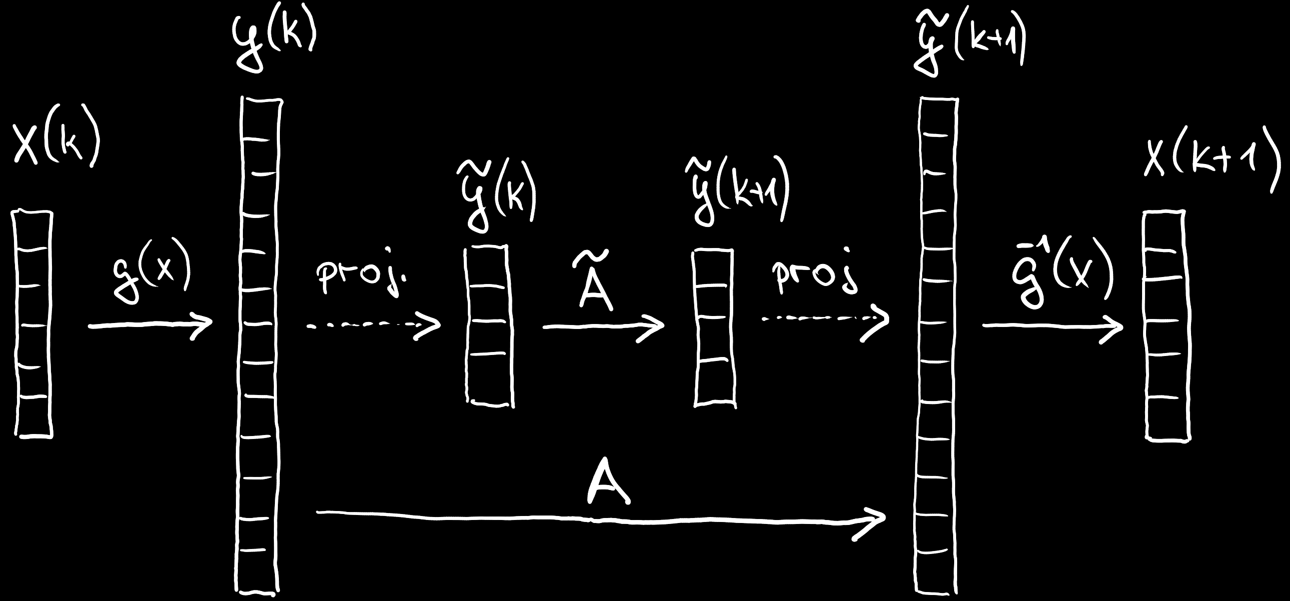
state variables x included in y.

Another example: $x_{k+1} = \beta x_k (1 - x_k)$

here even truncation doesn't perform well.

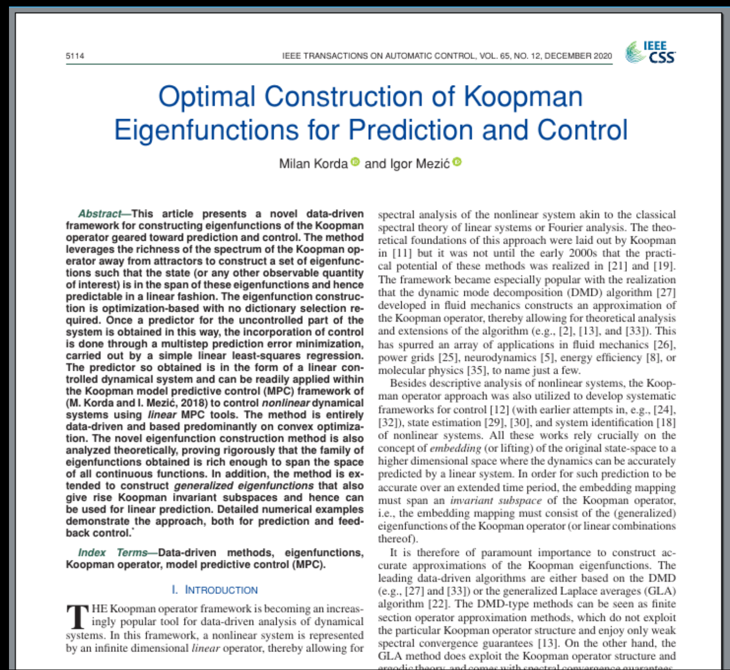
(Approximate) data-driven Koopman modelling

using a selection of observables = eDMD



How many and which $g(x)$?

Korda (2020)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Optimal Construction of Koopman Eigenfunctions for Prediction and Control

Milan Korda and Igor Mezić

Abstract*—This article presents a novel data-driven framework for constructing eigenfunctions of the Koopman operator geared toward prediction and control. The method leverages the richness of the spectrum of the Koopman operator away from attractors to construct a set of eigenfunctions such that the state (or any other observable quantity of interest) is in the span of these eigenfunctions and hence predictable in a linear fashion. The eigenfunction construction is optimization-based with no dictionary selection required. Once a predictor for the uncontrolled part of the system is obtained in this way, the incorporation of control is done through a multistep prediction error minimization, carried out by a simple linear least-squares regression. The predictor so obtained is in the form of a linear controlled dynamical system and can be readily applied within the Koopman model predictive control (MPC) framework of (M. Korda and I. Mezić, 2018) to control *nonlinear* dynamical systems using *linear* MPC tools. The method is entirely data-driven and based predominantly on convex optimization. The novel eigenfunction construction method is also analyzed theoretically, proving rigorously that the family of eigenfunctions obtained is rich enough to span the space of all continuous functions. In addition, the method is extended to construct *generalized eigenfunctions* that also give rise Koopman invariant subspaces and hence can be used for linear prediction. Detailed numerical examples demonstrate the approach, both for prediction and feedback control.

***Index Terms*—Data-driven methods, eigenfunctions, Koopman operator, model predictive control (MPC).**

I. Introduction

THE Koopman operator framework is becoming an increasingly popular tool for data-driven analysis of dynamical systems. In this framework, a nonlinear system is represented by an infinite dimensional *linear* operator, thereby allowing for spectral analysis of the nonlinear system akin to the classical spectral theory of linear systems or Fourier analysis. The theoretical foundations of this approach were laid out by Koopman in [11] but it was not until the early 2000s that the practical potential of these methods was realized in [21] and [19]. The framework became especially popular with the realization that the dynamic mode decomposition (DMD) algorithm [27] developed in fluid mechanics constructs an approximation of the Koopman operator, thereby allowing for theoretical analysis and extensions of the algorithm (e.g., [2], [13], and [33]). This has spurred an array of applications in fluid mechanics [26], power grids [25], neurodynamics [5], energy efficiency [8], or molecular physics [35], to name just a few.

Besides descriptive analysis of nonlinear systems, the Koopman operator approach was also utilized to develop systematic frameworks for control [12] (with earlier attempts in, e.g., [24], [32]), state estimation [29], [30], and system identification [18] of nonlinear systems. All these works rely crucially on the concept of *embedding* (or lifting) of the original state-space to a higher dimensional space where the dynamics can be accurately predicted by a linear system. In order for such prediction to be accurate over an extended time period, the embedding mapping must span an *invariant subspace* of the Koopman operator, i.e., the embedding mapping must consist of the (generalized) eigenfunctions of the Koopman operator (or linear combinations thereof).

It is therefore of paramount importance to construct accurate approximations of the Koopman eigenfunctions. The leading data-driven algorithms are either based on the DMD (e.g., [27] and [33]) or the generalized Laplace averages (GLA) algorithm [22]. The DMD-type methods can be seen as finite section operator approximation methods, which do not exploit the particular Koopman operator structure and enjoy only weak spectral convergence guarantees [13]. On the other hand, the GLA method does exploit the Koopman operator structure and

EIGENFUNCTIONS of Koopman operator

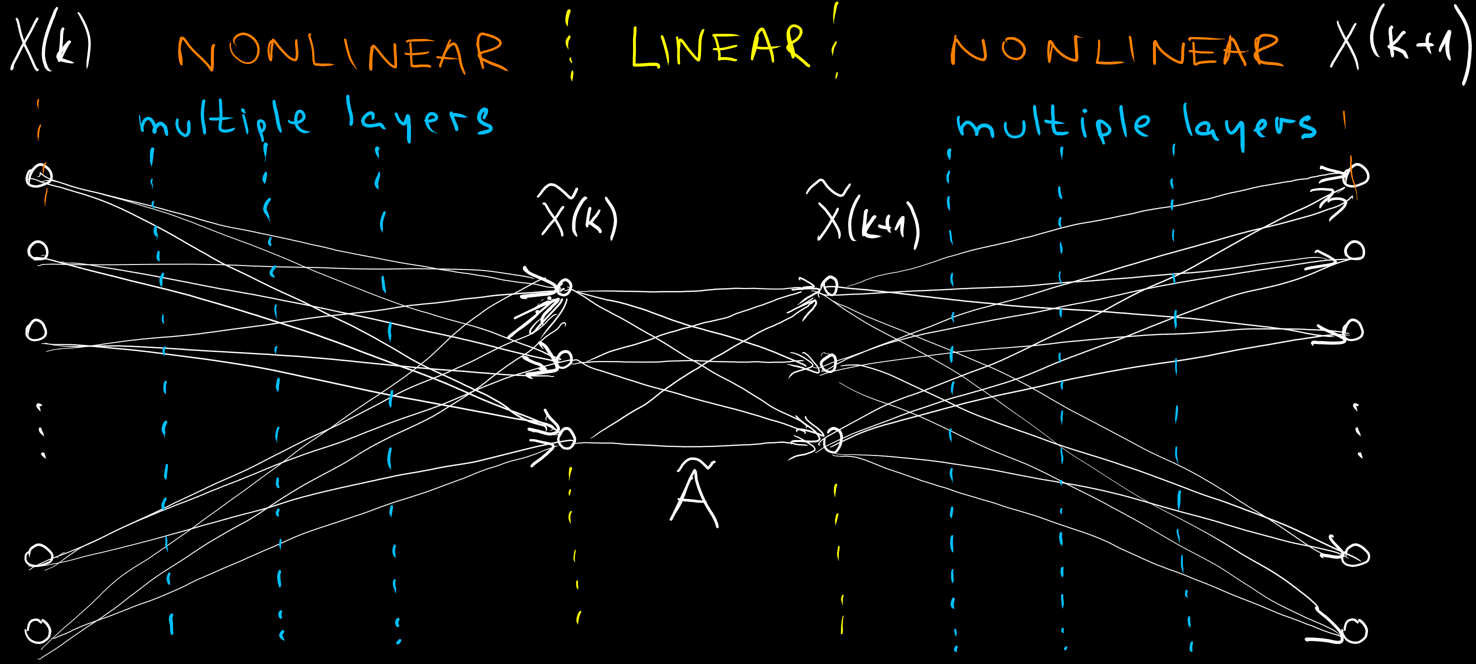
$$\boxed{\varphi(x_{k+1})} = \varphi\left(F_{\Delta t}(x_k)\right) = \left(\mathcal{K}_{\Delta t}\varphi\right)(x_k) = \boxed{\lambda \cdot \varphi(x_k)}$$

$$\boxed{\varphi(x_k) = \lambda^k \cdot \varphi(x_0)}$$

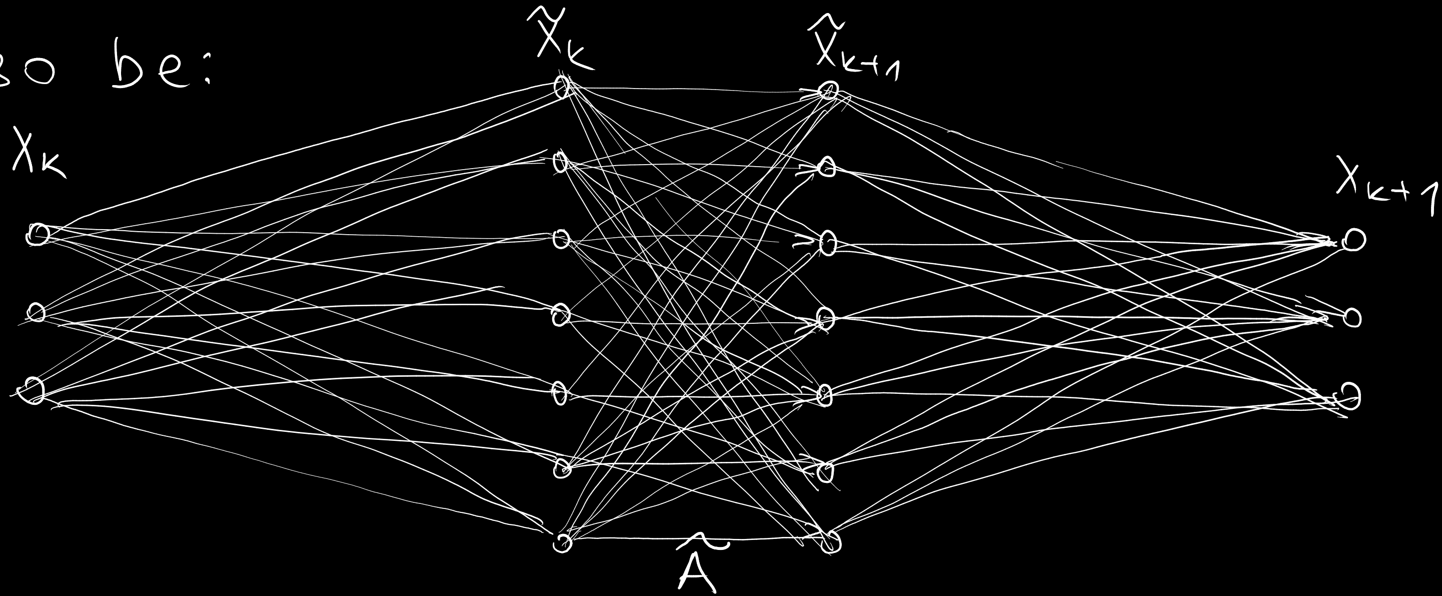
similarly in continuous time $\boxed{\dot{\varphi}(x(t)) = \lambda \cdot \varphi(x(t))}$

Generally, infinite # of eigenfunctions: $\varphi_1(\cdot), \varphi_2(\cdot), \ldots$

eDMD vs Deep Neural Nets



can also be:



Control using Koopman

- DMD_c Proctor (2016)
- eDMD Korda (2018)

Matlab code at

https://github.com/MilanKorda/KoopmanMPC

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Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control☆

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ABSTRACT

This paper presents a class of linear predictors for nonlinear controlled dynamical systems. The basic idea is to lift (or embed) the nonlinear dynamics into a higher dimensional space where its evolution is approximately linear. In an uncontrolled setting, this procedure amounts to numerical approximations of the Koopman operator associated to the nonlinear dynamics. In this work, we extend the Koopman operator to controlled dynamical systems and apply the Extended Dynamic Mode Decomposition (EDMD) to compute a finite-dimensional approximation of the operator in such a way that this approximation has the form of a linear controlled dynamical system. In numerical examples, the linear predictors obtained in this way exhibit a performance superior to existing linear predictors such as those based on local linearization or the so called Carleman linearization. Importantly, the procedure to construct these linear predictors is completely data-driven and extremely simple – it boils down to a nonlinear transformation of the data (the lifting) and a linear least squares problem in the lifted space that can be readily solved for large data sets. These linear predictors can be readily used to design controllers for the nonlinear dynamical system using linear controller design methodologies. We focus in particular on model predictive control (MPC) and show that MPC controllers designed in this way enjoy computational complexity of the underlying optimization problem comparable to that of MPC for a linear dynamical system with the same number of control inputs and the same dimension of the state-space. Importantly, linear inequality constraints on the state and control inputs as well as nonlinear constraints on the state can be imposed in a linear fashion in the proposed MPC scheme. Similarly, cost functions nonlinear in the state variable can be handled in a linear fashion. We treat both the full-state measurement case and the input–output case, as well as systems with disturbances/noise. Numerical examples demonstrate the approach.[1]

1. Introduction

This paper presents a class of linear predictors for nonlinear controlled dynamical systems. By a predictor, we mean an artificial dynamical system that can predict the future state (or output) of a given nonlinear dynamical system based on the measurement of the current state (or output) and given the current and future inputs of the system. We focus on predictors possessing a *linear*

The key step in obtaining accurate predictions of a nonlinear dynamical system as the output of a linear dynamical system is *lifting* of the state-space to a higher dimensional space, where the evolution of this lifted state is (approximately) linear. For uncontrolled dynamical systems, this idea can be rigorously justified using the Koopman operator theory (Mezić & Banaszuk, 2004; Mezić, 2005). The Koopman operator is a *linear* operator

Implementations of DMD/Koopman

Matlab :

Python :

Julia :