Model and controller order reduction

Graduate course on Optimal and Robust Control

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   - Frequency weighted approximation
   - Controller order reduction

Motivation

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  - Design a controller for the full-order model and then approximate the controller with a reduced-order controller
    Controller-order reduction cannot be done just by applying some technique for model-order reduction to a full controller
  - (Design a low-order controller directly)

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Useful in their own right (modeling and simulation).

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```
load clown
size(X)

figure(1)
image(X)
colormap(map)
```

```
[U,S,V] = svd(X);
k = 10
Xred = U(:,1:k)*S(1:k,1:k)*V(:,1:k)';

figure(2)
image(Xred)
colormap(map)
```

```
s = diag(S)

figure(3)
plot(s,'.')
ylabel('Singular values')
```

Bounds on the error (Schmidt-Eckart-Young-Mirsky)

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provided $\sigma_i > \sigma_{i+1}$.

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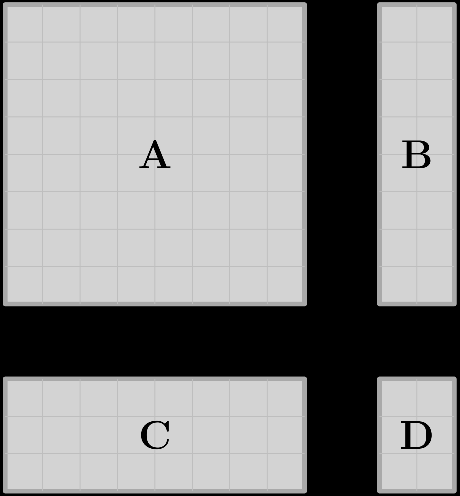
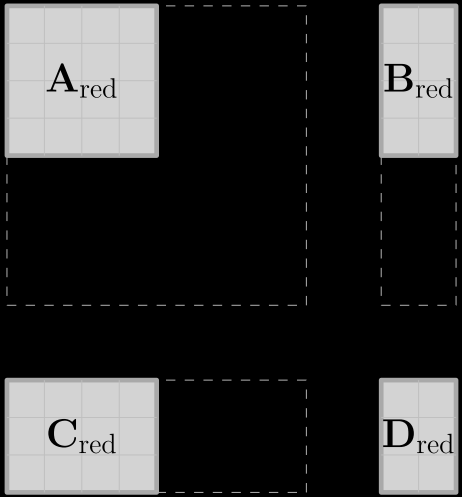
A nonunique minimizer is obtained by truncating the dyadic decomposition

$$X_{\min} = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots + \sigma_k u_k v_k^T$$


A
B
C
D



$\mathbf{A}_{\text{red}}$
$\mathbf{B}_{\text{red}}$
$\mathbf{C}_{\text{red}}$
$\mathbf{D}_{\text{red}}$

Full order model

$$\begin{aligned}
\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\
\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \\
y(t) &= C_1x_1(t) + C_2x_2(t) + Du(t)
\end{aligned}$$

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Reduced model (assuming $x_2 = 0$, relabelling x_1 as x)

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Matlab (Control System Toolbox): **modred(G,ix,'Truncate')**

Also *singular perturbation*. Assume

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from which (assuming A_{22} is nonsingular), we express x_2 and substitute to the first state equation

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$$\begin{aligned}\dot{x}_1(t) &= A_r x_1(t) + B_r u(t)\\ y(t) &= C_r x_1(t) + D_r u(t)\end{aligned}$$

where

$$\begin{aligned}A_r &= A_{11} - A_{12}A_{22}^{-1}A_{21}\\ B_r &= B_1 - A_{12}A_{22}^{-1}B_2\\ C_r &= C_1 - C_2A_{22}^{-1}A_{21}\\ D_r &= D - C_2A_{22}^{-1}B_2\end{aligned}$$

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Matlab: **modred(G,ix,'MatchdDC')**

$\boxed{\text{Both truncation and residualization depend on the chosen basis}}$

Modal truncation

Diagonal (modal) realization

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General heuristic rule: eliminating the fastest modes, hence truncating the corresponding state variables.

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General heuristic rule: eliminating the fastest modes, hence truncating the corresponding state variables.

```
w(1) = 0.56806689746895; zeta(1) = 0.00096819582773; k(1) = 0.01651378989774;
w(2) = 3.94093897440699; zeta(2) = 0.00100229920475; k(2) = 0.00257034576009;
w(3) = 10.58229653714164;zeta(3) = 0.00100167293203; k(3) = 0.00002188016252;
w(4) = 16.19234386986640;zeta(4) = 0.01000472824082; k(4) = 0.00027927762861;

G = ss(0);
for i=1:4
    Gi = k(i)*tf(w(i)^2,[1,2*zeta(i)*w(i),w(i)^2]);
    G = G + ss(Gi);
end
```

```
Gmt = modred(G,5:8,'Truncate')
```

Modal truncation caveats

Generally works fine, but sometimes fast modes can affect the input-output behaviour more than the low frequency ones

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```
k(1) = 0.01; k(2) = 0.005; k(3) = 0.007; k(4) = 0.004;

G = ss(0);
for i=1:4
    Gi = k(i)*tf(w(i)^2,[1,2*zeta(i)*w(i),w(i)^2]);
    G = G + ss(Gi);
end
```

```
Gmt = modred(G,5:8,'Truncate')          % Try 'MatchDC' option too
```

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$$\mathsf{A}\mathsf{V} = \mathsf{V}\mathsf{D}, \qquad \mathsf{A} = \mathsf{V}\mathsf{D}\mathsf{V}^{-1}$$

$$\mathsf{Av} = \lambda \mathsf{v}$$

λ is an eigenvalue, v is an associated (right) eigenvector.

$$\mathsf{AV} = \mathsf{VD}, \qquad \mathsf{A} = \mathsf{VDV}^{-1}$$

Similarly the left version

$$\mathsf{w}^{\mathrm{T}}\mathsf{A} = \mathsf{w}^{\mathrm{T}}\lambda$$

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$$\mathsf{W}^{\mathrm{T}}\mathsf{A} = \mathsf{D}\mathsf{W}^{\mathrm{T}}, \qquad \mathsf{A}^{\mathrm{T}}\mathsf{W} = \mathsf{WD}$$

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Similarly the left version

$$\mathsf{w}^{\mathrm{T}}\mathsf{A} = \mathsf{w}^{\mathrm{T}}\lambda$$

$$\mathsf{W}^{\mathrm{T}}\mathsf{A} = \mathsf{DW}^{\mathrm{T}}, \qquad \mathsf{A}^{\mathrm{T}}\mathsf{W} = \mathsf{WD}$$

$$\mathsf{w}_i^{\mathrm{T}}\mathsf{v}_j = 0, \qquad \mathsf{w}_i^{\mathrm{T}}\mathsf{v}_i = 1, \qquad i \neq j.$$

Matlab: **eig()**

```
>> A = [1, 2, 3; 4, 5, 6; 7, 8, 9]
A =
     1     2     3
     4     5     6
     7     8     9
>> [V,D,W] = eig(A)
V =
   -0.2320   -0.7858    0.4082
   -0.5253   -0.0868   -0.8165
   -0.8187    0.6123    0.4082
D =
   16.1168         0         0
         0   -1.1168         0
         0         0   -0.0000
W =
   -0.4645   -0.8829    0.4082
   -0.5708   -0.2395   -0.8165
   -0.6770    0.4039    0.4082
```

$$e^{\mathsf{A}t} = \mathsf{A}_1 e^{\lambda_1 t} + \mathsf{A}_2 e^{\lambda_2 t} + \ldots + \mathsf{A}_n e^{\lambda_n t}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A (for $\lambda_i \neq \lambda_j$)

$$e^{At} = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \ldots + A_n e^{\lambda_n t}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A (for $\lambda_i \neq \lambda_j$)
or

$$e^{At} = A_1 e^{\lambda_1 t} + A_2 t e^{\lambda_1 t} + A_3 e^{\lambda_3 t} + \ldots + A_n e^{\lambda_n t}$$

for $\lambda_1 = \lambda_2$,

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$$\mathsf{x}(t) = e^{\mathsf{A}t}\mathsf{x}(0) = \mathsf{A}_1 e^{\lambda_1 t}\mathsf{x}(0) + \mathsf{A}_2 e^{\lambda_2 t}\mathsf{x}(0) + \ldots + \mathsf{A}_n e^{\lambda_n t}\mathsf{x}(0).$$

Solution of state equations, Matrix exponential, Modes

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$$\boxed{\mathsf{x}(t) = \mathsf{v}_1 e^{\lambda_1 t} \underbrace{\mathsf{w}_1^{\mathrm{T}}\mathsf{x}(0)}_{\alpha_1} + \mathsf{v}_2 e^{\lambda_2 t} \underbrace{\mathsf{w}_2^{\mathrm{T}}\mathsf{x}(0)}_{\alpha_2} + \ldots + \mathsf{v}_n e^{\lambda_n t} \underbrace{\mathsf{w}_n^{\mathrm{T}}\mathsf{x}(0)}_{\alpha_n}.}$$

$$y(t) = Cx(t)$$

$$\mathsf{y}(t) = \mathsf{C}\mathsf{x}(t)$$

In the single-output case

$$\boxed{y(t) = \underbrace{\mathsf{C}\mathsf{v}_1\mathsf{w}_1^{\mathrm{T}}\mathsf{x}(0)}_{\gamma_1} e^{\lambda_1 t} + \underbrace{\mathsf{C}\mathsf{v}_2\mathsf{w}_2^{\mathrm{T}}\mathsf{x}(0)}_{\gamma_2} e^{\lambda_1 t} + \ldots + \underbrace{\mathsf{C}\mathsf{v}_n\mathsf{w}_n^{\mathrm{T}}\mathsf{x}(0)}_{\gamma_n} e^{\lambda_n t}.}$$

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In Laplace domain

$$Y(s) = \frac{\gamma_1}{s-\lambda_1} + \frac{\gamma_2}{s-\lambda_2} + \ldots + \frac{\gamma_n}{s-\lambda_n}.$$

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Directional analysis (through eigenvalue decomposition):

$$Q = U_Q \Sigma_Q U_Q^*, \qquad Q = U_Q \Sigma_Q U_Q^T \text{ for real eigenvectors}$$

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Continuous-time Lyapunov matrix

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Matlab: **lyap()**, **gram()**

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$\mathsf{P} = \mathsf{P}^{\mathrm{T}} \in \mathbb{R}^{n \times n}$, $\mathsf{P} \geq 0$; For (A, B) stable and reachable/controllable $\mathsf{P} > 0$.

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Recall

$$\mathsf{x}(t) = e^{\mathsf{A}t}\mathsf{x}(0) + \int_0^t e^{\mathsf{A}(t-\tau)}\mathsf{B}\mathsf{u}(\tau)d\tau$$

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Minimum energy signal $\mathsf{u}_{\mathsf{min}}$ needed to bring system to a given state $\mathsf{x}(0) = \mathsf{x}_{\mathrm{r}}$

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Minimum energy signal $\mathsf{u}_{\mathsf{min}}$ needed to bring system to a given state $\mathsf{x}(0) = \mathsf{x}_\mathrm{r}$ has the energy

$$\boxed{\|\mathsf{u}_{\mathsf{min}}\|_2^2 = \mathsf{x}_\mathrm{r}^\mathsf{T} \mathsf{P}^{-1} \mathsf{x}_\mathrm{r}}$$

$$P = \int_0^\infty \underbrace{e^{At}B}_{\text{input-to-state map}} B^T e^{A^T t} dt$$

$P = P^T \in \mathbb{R}^{n \times n}$, $P \geq 0$; For (A, B) stable and reachable/controllable $P > 0$.
Recall

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Minimum energy signal u_{min} needed to bring system to a given state $x(0) = x_r$ has the energy

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Do the same directionality analysis using $P = U_P \Sigma_P U_P^*$

Computation of the (infinite) reachability gramian

Continuous-time Lyapunov equation

$$\mathsf{AP} + \mathsf{PA}^\mathsf{T} + \mathsf{BB}^\mathsf{T} = 0$$

Matlab: **lyap()**
System reachable iff $\mathsf{P} > 0$.
Discrete-time Lyapunov equation

$$\mathsf{APA}^\mathsf{T} + \mathsf{BB}^\mathsf{T} = \mathsf{P}$$

Matlab: **dlyap()**

Example in Matlab

1. Different modes exhibit themselves in different directions in the state space (they are differently projected into the state variables).

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Simple truncation or residualization will not work well. So what?

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Matlab: finding Hankel singular values with **hsvd()** (Control System Tbx) or **hankelsv()** (Robust Control Tbx)

Algorithm for balancing transformation

Two key computations:

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Cholesky decomposition of P

$$\boxed{P = LL^*}$$

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Diagonalizing similarity transformation $TPQT^{-1} = \bar{P}\bar{Q} = \Sigma^2$ but also $\bar{P} = \Sigma$ and $\bar{Q} = \Sigma$, is

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Matlab (Control System Tbx): **balreal(), balred()**

Properties of balanced and truncated model

Reduced (truncated) system given by (A_{11}, B_1, C_1, D) is

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(without multiplicities!)

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(without multiplicities!)

However, **not minimizing the error!**

Matlab example

Need for more efficient/reliable balancing algorithms

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More in

1. ANTOULAS, Athanasios C. Approximation of large-scale dynamical systems. SIAM, 2005.
2. ANTOULAS, Athanasios C.; SORENSEN, Dan C. Approximation of large-scale dynamical systems: An overview. 2001. Download at `https://scholarship.rice.edu/bitstream/handle/1911/101964/TR01-01.pdf?sequence=1`

We need three matrix decompositions

Upper triangular Cholesky decomposition of P

$$P = UU^*$$

Lower triangular Cholesky decomposition of Q

$$Q = LL^*$$

Singular value decomposition of U*L

$$U^*L = W\Sigma V^*$$

Square root balancing (and truncating)

Product of gramians can be then expressed as

$$\begin{aligned}
\mathsf{PQ} &= \mathsf{UU^*LL^*} \\
&= \mathsf{U(U^*L)(L^*U)U^{-1}} \\
&= \mathsf{U(U^*L)(V\Sigma W^*)U^{-1}}
\end{aligned}$$

Diagonalizing similarity transformation is

$$\mathsf{T} = \Sigma^{-1/2}\mathsf{V^*L^*}, \quad \mathsf{T}_i = \underbrace{\mathsf{UW}\Sigma^{-1/2}}_{\mathsf{T}^{-1}}$$

Finding only the relevant part of the transformation matrix

$$\mathsf{T}_1 = \Sigma_1^{-1/2}\mathsf{V}_1^*\mathsf{L}^*, \quad \mathsf{T}_{1i} = \mathsf{UW}_1\Sigma_1^{-1/2}$$

Truncated system given by $(\mathsf{T}_1\mathsf{AT}_{1i}, \mathsf{T}_1\mathsf{B}, \mathsf{CT}_{1i}, \mathsf{D})$.
Matlab (Robust Control Tbx): **balancmr()**

Square root balancing up to scaling

Simply omit the scaling $\Sigma^{-1/2}$. Can improve conditioning. Scale the system instead.

$$T = V^* L^*, \quad T_i = UW$$

$$\left[\begin{array}{c|c} \Sigma^{-1/2} A \Sigma^{-1/2} & \Sigma^{-1/2} B \\ \hline C \Sigma^{-1/2} & D \end{array}\right]$$

Two extra matrix decompositions needed

QR decomposition of UW

$$\mathsf{UW} = \mathsf{X}\Phi$$

QR decomposition of LV

$$\mathsf{LV} = \mathsf{Y}\Psi$$

where X, Y are orthogonal, Φ, Ψ are upper triangular.

Balancing free square root algorithm (A.Varga, 1991)

Transformation diagonalizing up to a triangular transformation is given by

$$\mathsf{T} = (\mathsf{Y}^*\mathsf{X})^{-1}\mathsf{Y}^* = \mathsf{X}^*, \quad \mathsf{T}^{-1} = \mathsf{X}$$

Then

$$\left[\begin{array}{c|c} \mathsf{X}^*\mathsf{A}\mathsf{X} & \mathsf{X}^*\mathsf{B} \\ \hline \mathsf{C}\mathsf{X} & \mathsf{D} \end{array}\right]$$

is balanced up to an upper triangular matrix $\mathsf{K} = \Sigma^{1/2}\Phi$.
Next truncate QR factorizations of UW and LW

$$\mathsf{U}\mathsf{W}_1 = \mathsf{X}_k\Phi_k, \quad \mathsf{L}\mathsf{V}_1 = Y_k\Psi_k$$

Then

$$\mathsf{T}_1 = (\mathsf{Y}_k^*\mathsf{X}_k)^{-1}\mathsf{Y}_k^*, \quad \mathsf{T}_{1i} = \mathsf{X}_k$$

transforms $(1,1)$ subsystem to

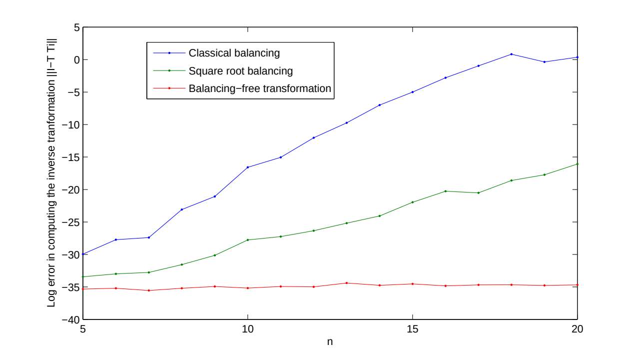
$$\left[\begin{array}{c|c} \mathsf{T}_1\mathsf{A}\mathsf{T}_{1i} & \mathsf{T}_1\mathsf{B} \\ \hline \mathsf{C}\mathsf{T}_{1i} & \mathsf{D} \end{array}\right]$$

Matlab: **balred()**

Numerical comparison of various ways of balancing

```
[A,B,C,D] = butter(n,1,'s');
```

The three algorithms for computing similarity transformation are compared w.r.t. $\|I - TT_i\|_2$



Nondiagonalizing (triangularizing) similarity transf. of PQ

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Schur decomposition of PQ in ascending order

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Keep the columns of V_A and V_D corresponding to the k large eigs

$$\mathsf{V}_a = \mathsf{V}_A \begin{bmatrix} 0 \\ \mathsf{I}_k \end{bmatrix}, \quad \mathsf{V}_d = \mathsf{V}_D \begin{bmatrix} \mathsf{I}_k \\ 0 \end{bmatrix}$$

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Singular value decomposition of $\mathsf{V}_a^*\mathsf{V}_d$

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The transformation matrices are

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Matlab: **schurmr()**

Different objects can be used for balancing. Instead of solution to Lyapunov eqs, solutions to some other equations can be used: Riccati equations?

Assumptions

- square systems
- stable
- D nonsingular

System transfer function

$$\mathrm{H}(s) = \mathrm{C}(s\mathrm{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D}$$

Reachability gramian P

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$$A^TQ + QA + (C - \bar{B}^TQ)^T(DD^T)^{-1}(C - \bar{B}^TQ) = 0$$

Finding a diagonalizing transformation for PQ and applying to the original system.

Balanced stochastic truncation

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Stochastic balancing guarantees multiplicative error bounds:

$$\sigma_{k+1} \leq \|\mathsf{H}^{-1}(\mathsf{H} - \mathsf{H}_k)\|_\infty \leq \prod_{i=k+1}^{n} \frac{1+\sigma_i}{1-\sigma_i} - 1$$

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Matlab: **bstmr()**

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Given a system G find a reduced-order G_r minizing the error as in

$$\min_{\mathsf{G}_r} \|\mathsf{G} - \mathsf{G}_r\|_H$$

Hankel operator Γ_G (for a given system G) maps past inputs into future outputs

$$\mathrm{y}(t) = \int_{-\infty}^{0} \underbrace{\mathrm{C}e^{\mathrm{A}(t-\tau)}\mathrm{B}}_{\mathrm{h}(t-\tau)} \mathrm{u}(\tau)d\tau, \quad t > 0$$

Hankel operator, Hankel norm

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$$\boxed{\|\mathrm{G}\|_H = \sqrt{\rho(\mathrm{QP})}}$$

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Apparently,

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$$\boxed{\|G\|_H = \sqrt{\rho(QP)}}$$

Matlab: **hankelmr()**

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$$\|W_o(G - G_r)W_i\|_\infty$$

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Different focus at different frequencies. Find a reduced-order system such that

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For two-sided case: no bound, no stability guarantees (refutation of Enn's conjecture)

Make

$$\|(G - G_r)V\|_\infty$$

small or at least guaranteed.
With the state space models of G and V

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \quad V = \left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right]$$

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small or at least guaranteed.
With the state space models of G and V

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \quad V = \left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right]$$

The state-space realization of GV is

$$GV = \left[\begin{array}{cc|c} A & BC_v & BD_v \\ 0 & A_v & B_v \\ \hline C & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

(New) controllability gramian

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Then... Balancing the product PQ... Stability guaranteed. Some ugly upper bounds available...

Replace the original full-order K with the reduced-order K_{red}

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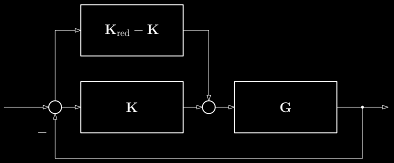

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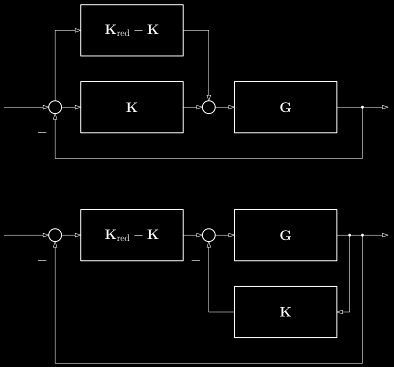

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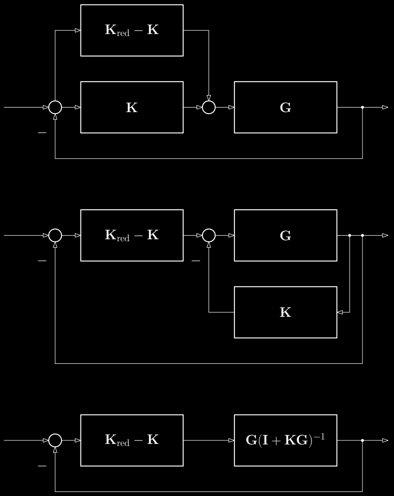

Sufficient conditions for K_{red} to be stabilizing

- K and K_{red} have the same number of unstable poles and $K - K_{\mathrm{red}}$ is bounded on the imaginary axis

Sufficient conditions for $\mathsf{K}_{\mathrm{red}}$ to be stabilizing

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$$\|(\mathsf{K} - \mathsf{K}_{\mathrm{red}})\mathsf{G}(\mathsf{I} + \mathsf{KG})^{-1}\|_\infty < 1$$

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Hence *frequency weighting matrix*

$$\mathsf{V}(s) = \mathsf{G}(\mathsf{I} + \mathsf{KG})^{-1} = (\mathsf{I} + \mathsf{GK})^{-1}\mathsf{G}$$