Model predictive control (MPC) for MLD systems

Via mixed integer programming (MILP or MIQP)

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Optimal control on a finite horizon

$$\underset{u_0,u_1,\ldots,u_{N-1}}{\text{minimize}}\ J_0(x(0),(u_0,u_1,\ldots,u_{N-1}))$$

subject to

$$\begin{aligned}
x_{k+1} &= Ax_k + B_u u_k + B_\delta \delta_k + B_z z_k + B_0\\
y_k &= Cx_k + D_u u_k + D_\delta \delta_k + D_z z_k + D_0\\
E_\delta \delta_k + E_z z_k &\le E_u u_k + E_x x_k + E_0\\
u_{\min} &\le u_k \le u_{\max}\\
x_{\min} &\le x_k \le x_{\max}\\
P x_N &\le r\\
x_0 &= x(0)
\end{aligned}$$

Cost function

- Weighted norm of the state and the control input.
- The norm can be 2-norm, 1-norm, infinity-norm, …

$$J_0(x(0), U_0) = x_N^T S_N x_N + \sum_{k=0}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right)$$

$$J_0(x(0), U_0) = \|S_N x_N\|_1 + \sum_{k=0}^{N-1} (\|Q x_k\|_1 + \|R u_k\|_1)$$

$$J_0(x(0), U_0) = \|S_N x_N\|_\infty + \sum_{k=0}^{N-1} (\|Q x_k\|_\infty + \|R u_k\|$$

Explicit MPC

- No online optimization needed.
- The controller is a piecewise affine function of the state (and reference)

$$
u(x,r) = \begin{cases} F_1 x + E_1 r + g_1 & \text{if } x \in H_1 \begin{bmatrix} x \\ u \end{bmatrix} \leq \\ & \vdots \\ F_m x + E_m r + g_m & \text{if } x \in H_m \begin{bmatrix} x \\ u \end{bmatrix} \leq \end{cases}
$$

- Based on multi-parametric programming.

Multiparametric programming

$$J^*(x) = \inf_z J(z,x)$$

- z … optimization variable
- x … parameter
- We study how the optimal cost depends on the parameter -> parametric programming.
- If x is a vector -> multiparametric programming.

Scalar example

$$
\begin{aligned}
J(x, z) &= \frac{1}{2} z^2 + 2zx + 2x^2 \\
\text{subject to} \quad & z \leq 1 + x.
\end{aligned}
$$

- Introduce Lagrange multiplicator

$$L(z, x, \lambda) = \frac{1}{2} z^2 + 2zx + 2x^2 + \lambda(z - 1 - x)$$

- KKT conditions

$$
\begin{aligned}
z + 2x + \lambda &= 0 \\
z - 1 - x &\leq 0 \\
\lambda &\geq 0 \\
\lambda(z - 1 - x) &= 0.
\end{aligned}
$$

- Two scenarios:
- $\lambda = 0$:

$$
\begin{aligned}
z + 2x &= 0 \\
z - 1 - x &\leq 0.
\end{aligned}
$$

$$
\begin{aligned}
z &= -2x \\
x &\geq -\frac{1}{3} \\
J^*(x) &= 0
\end{aligned}
$$

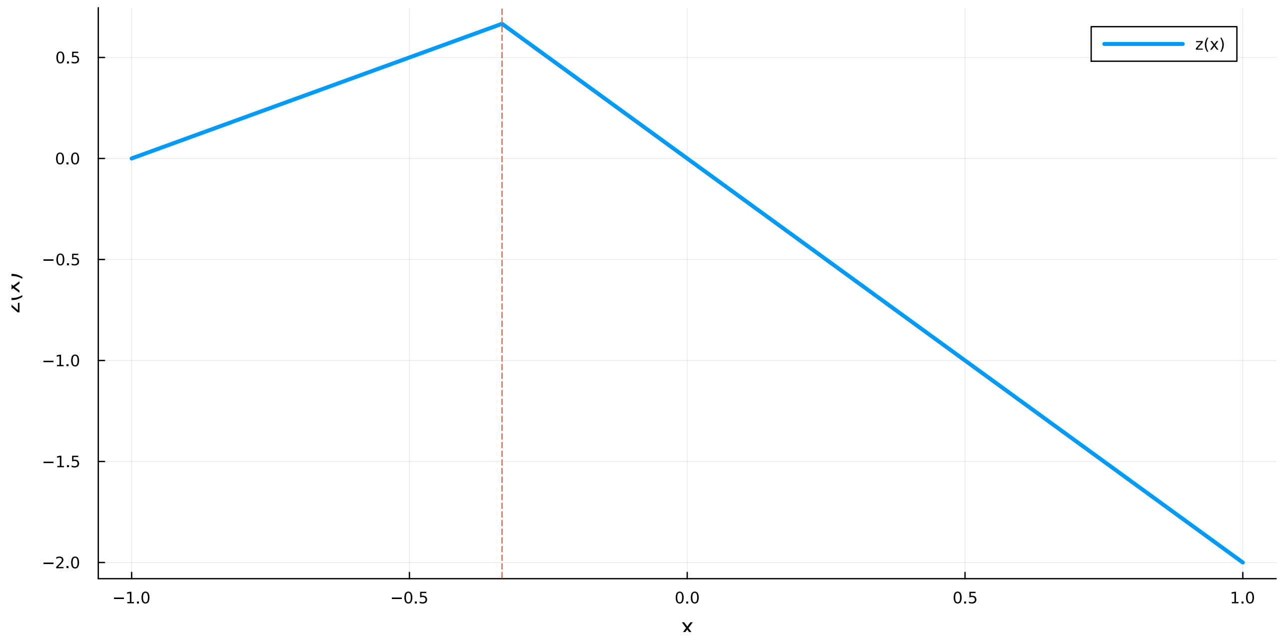
- $z - 1 - x = 0$:

$$
\begin{aligned}
z + 2x + \lambda &= 0 \\
z - 1 - x &= 0 \\
\lambda &\geq 0.
\end{aligned}
$$

$$
\begin{aligned}
z &= 1 + x \\
\lambda &= -z - 2x \geq 0 \qquad \Longrightarrow \qquad x \leq -\frac{1}{3} \\
J^*(x) &= \frac{9}{2}x^2 + 3x + \frac{1}{2}
\end{aligned}
$$

- The two scenarios can now be combined into a single piecewise affine function $z(x)$

$$
z(x) = \begin{cases} 1 + x & \text{if } x \leq -\frac{1}{3}, \\ -2x & \text{if } x > -\frac{1}{3}. \end{cases}
$$



- and a piecewise quadratic cost function $J^*(x)$

$$J^*(x) = \begin{cases} \frac{9}{2}x^2 + 3x + \frac{1}{2} & \text{if } x \leq -\frac{1}{3}, \\ 0 & \text{if } x > -\frac{1}{3}. \end{cases}$$