Solutions of hybrid systems

(Some more) concepts, conditions

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Hybrid time and hybrid time domain

- In continuous-time systems: continuous time $t \in \mathbb{R}_{\geq 0}$
- In discrete-time systems: discrete "time" $k \in \mathbb{N}$
- In hybrid systems: hybrid time (both cont. and disc. time)

$$(t, j),\ t \in \mathbb{R}_{\geq 0},\ j \in \mathbb{N}$$

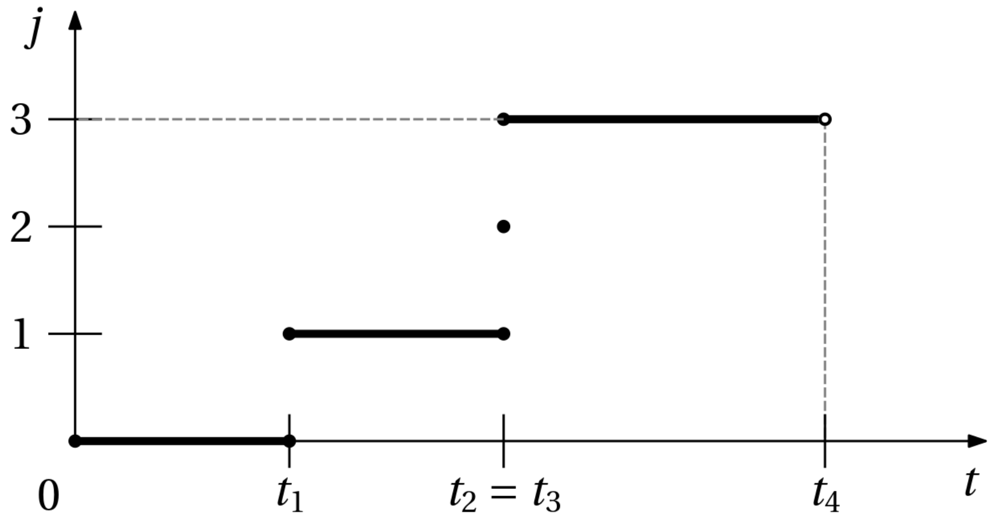
- Hybrid time domain

$$E \subset [0, T] \times \{0, 1, 2, \ldots, J\},$$

where T and J can be ∞.

- Particularly,

$$E = \bigcup_{j=0}^{J} \left([t_j, t_{j+1}] \times \{j\} \right)$$



- Totally ordered – we can decide if $(t, j) \leq (t', j')$ if both hybrid times are from the same hybrid domain.

Hybrid arc (hybrid state trajectory)

- As formulated within the framework of hybrid state equations:

$$x : E \to \mathbb{R}^n$$

- For each j the function $t \mapsto x(t, j)$ is absolutely continuous on the interval $I^j = \{t \mid (t, j) \in E\}$.
- Abusing the notation a bit, since normally we use $x(t)$ even within hybrid systems…
- Hybrid time domain is only known after knowing the solution (the arc, the trajectory). Compare this with the continuous-time or discrete-time system.

Hybrid input

- It has its own hybrid domain E_{u}.

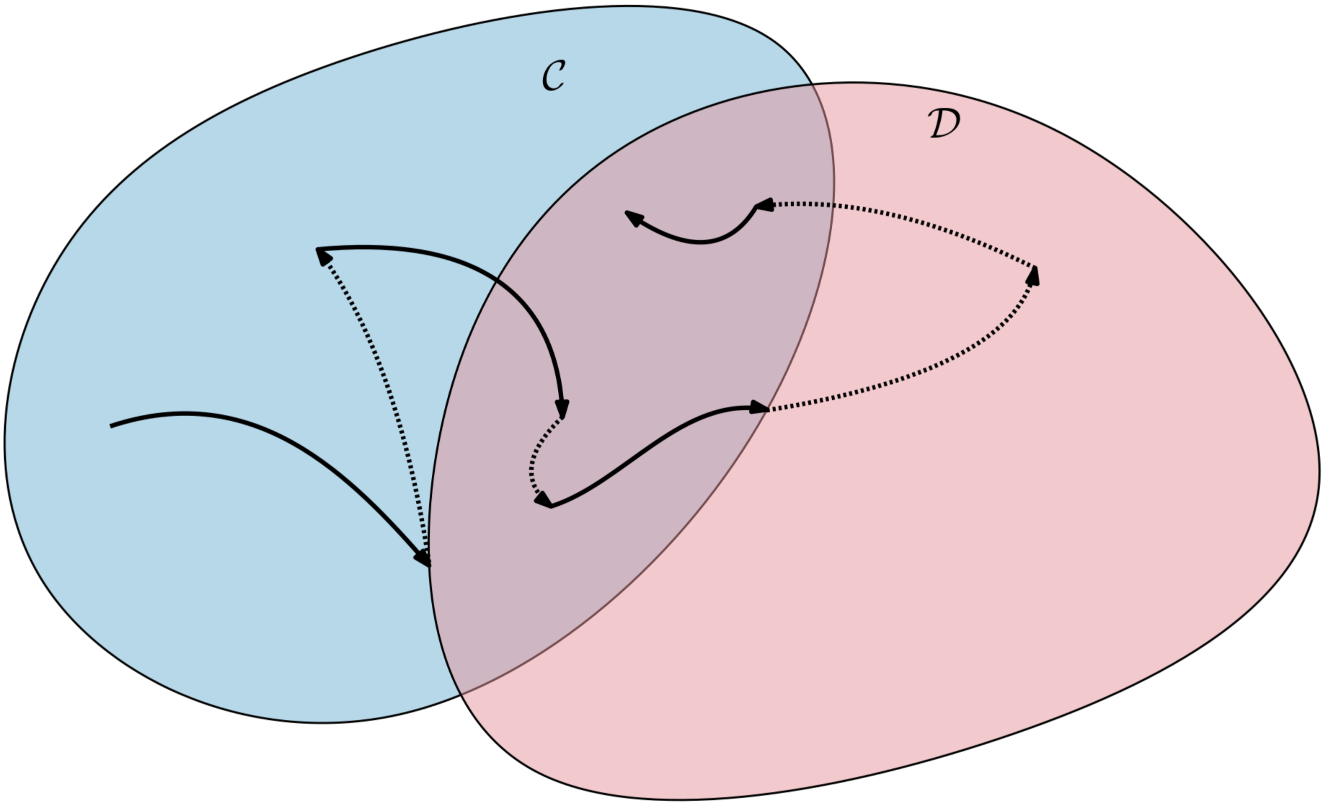
$$u : E_{\mathrm{u}} \to \mathbb{R}^m$$

- For each j the function $t \mapsto u(t,j)$ must be… well-behaved…
  - For example, piecewise continuous on the interval $I^j = \{t \mid (t,j) \in E_{\mathrm{u}}\}$.

Solutions of autonomous (no-input) systems

- A hybrid arc $x(\cdot,\cdot)$ is a solution to the hybrid equations given by the common quadruple $\{\mathcal{C},\mathcal{D},f,g\}$ (or $\{\mathcal{C},\mathcal{D},\mathcal{F},\mathcal{G}\}$), if
  - the initial state $x(0,0)\in\overline{\mathcal{C}}\cup\mathcal{D}$, and
  - for all j such that $I^j=\{t\mid(t,j)\in E\}$ has a nonempty interior $\operatorname{int} I^j$
    - $x(t,j)\in\mathcal{C}\;\forall t\in\operatorname{int} I^j$,
    - $\dot x(t,j)=f(x(t,j))\text{ for almost all }t\in I^j$, and
  - for all $(t,j)\in E$ such a $(t,j+1)\in E$
    - $x(t,j)\in\mathcal{D}$, and
    - $x(t,j+1)=g(x(t,j))$.
- Make the modifications for the $\{\mathcal{C},\mathcal{D},\mathcal{F},\mathcal{G}\}$ version by yourself.

Evolution of a solution



Solutions of systems with inputs

- Hybrid time domains for the arcs and inputs are the same.
- The initial state-control pair $(x(0,0), u(0,0)) \in \overline{\mathcal{C}} \cup \mathcal{D}$, and
  - for all j such that $I^j = \{t \mid (t,j) \in E\}$ has a nonempty interior $\operatorname{int} I^j$
    - $(x(t,j), u(t,j)) \in \mathcal{C}\ \forall t \in \operatorname{int} I^j$,
    - $\dot{x}(t,j) = f(x(t,j), u(t,j)) \text{ for almost all } t \in I^j$, and
  - for all $(t,j) \in E$ such a $(t, j+1) \in E$
    - $(x(t,j), u(t,j)) \in \mathcal{D}$, and
    - $x(t, j+1) = g(x(t,j), u(t,j))$.

Types of solutions

- based on their hybrid time domain E:
- trivial: just one point;
- nontrivial: at least two points;
- complete: if the domain is unbounded;
- bounded, compact: if the domain is bounded, compact;
- discrete: if nontrivial and $E \subset \{0\} \times \mathbb{N}$;
- continuous: if nontrivial and $E \subset \mathbb{R}_{\geq 0} \times \{0\}$;
- eventually discrete: if $T = \sup_E t < \infty$ and $E \cap (\{T\} \times \mathbb{N})$ contains at least two points;
- eventually continuous: if $J = \sup_E j < \infty$ and $E \cap (\mathbb{R}_{\geq 0} \times \{J\})$ contains at least two points;
- Zeno: if complete and $\sup_E t < \infty$;
- maximal

Maximal solution

- It cannot be extended.
- Solution $x(t,j)$ defined on the hybrid time domain E is maximal, if
  - on an extended hybrid time domain E^ext such that $E\subset E^\mathrm{ext}$,
  - there is no solution $x^\mathrm{ext}(t,j)$ that coincices with x on E.
- Some use the "linguistic" terminology that a maximal solution is not a prefix to any other solution.
- Complete solutions are maximal. But not vice versa.

Example of a (non-)maximal solution

$$\dot{x} = 1,\ x(0) = 1$$

$$(t, j) \in [0, 1] \times \{0\}$$

Now extend the time domain to

$$(t, j) \in [0, 2] \times \{0\}.$$

…

Maximal but not complete continuous solution?

- Finite escape time

$$\dot{x} = x^2,\ x(0) = 1,$$

$$x(t) = 1/(1-t)$$

- Discontinuous right hand side

$$\dot{x} = \begin{cases} -1 & x > 0 \\ 1 & x \leq 0 \end{cases}, \quad x(0) = -1$$

`(unless the concept of Filippov solution is invoked).`

Example of Zeno sol'n: bouncing ball

- Starting on the ground with some initial upward velocity

$$h(t) = \underbrace{h(0)}_{0} + v(0)t - \frac{1}{2}gt^2, \quad v(0) = 1$$

- What time will it hit the ground again?

$$0 = t - \frac{1}{2}gt^2 = t(1 - \frac{1}{2}gt)$$

$$t_1 = \frac{2}{g}$$

- Simplify (scale) the computations just to get the qualitative picture: set $g = 2$, which gives $t_1 = 1$.

$t_1 = 1$:

$$v(t_1^+) = \gamma v(t_1) = \gamma v(0) = \gamma$$

The next hit will be at $t_1 + \tau_1$

$$h(t_1 + \tau_1) = 0 = \gamma\tau_1 - \tau_1^2 = \tau_1(\gamma - \tau_1)$$

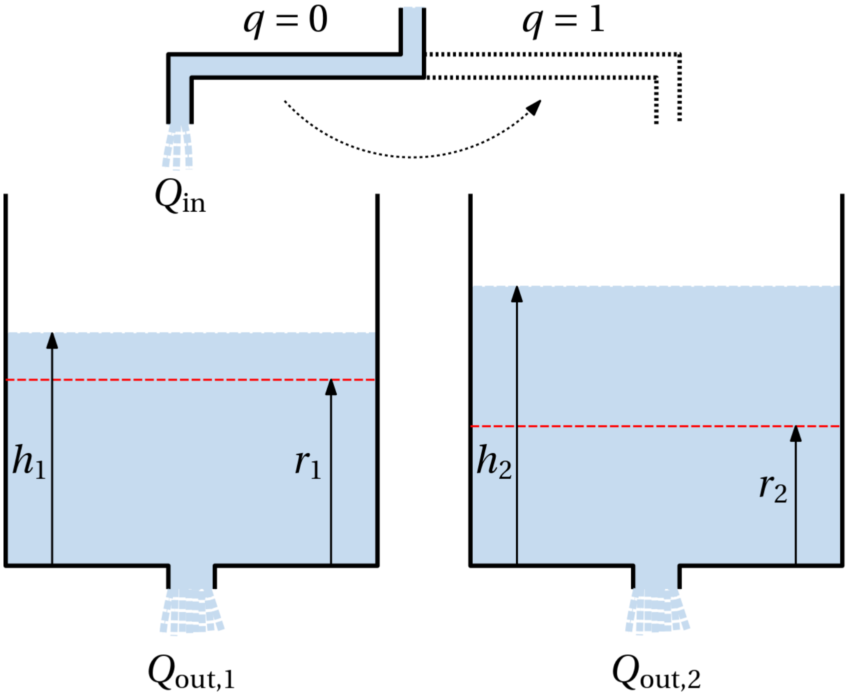
$$\tau_1 = \gamma$$

$$t_2 = t_1 + \tau_1 = 1 + \gamma : \quad \ldots$$
$$t_k = 1 + \gamma + \gamma^2 + \ldots + \gamma^k : \quad \ldots$$

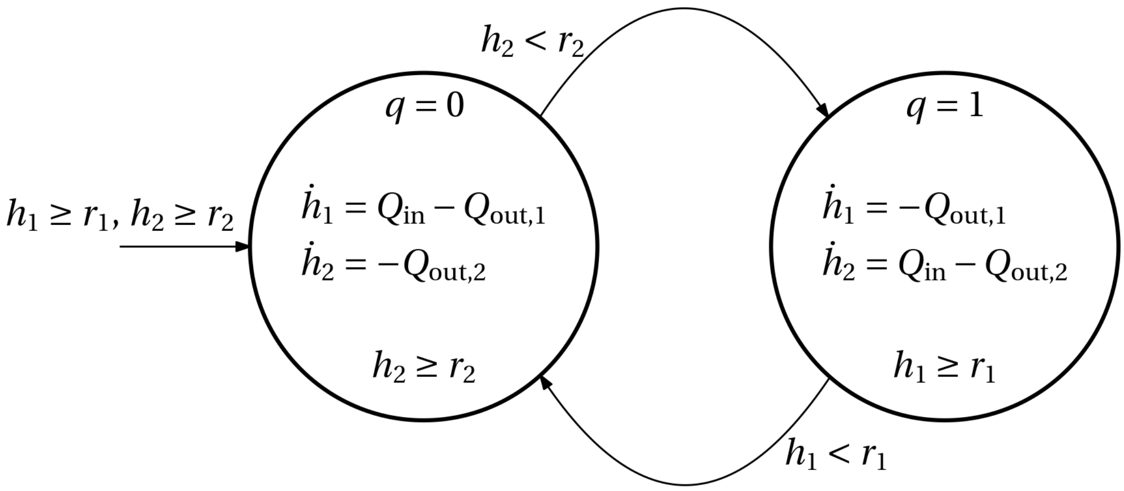
$$\boxed{\lim_{k \to \infty} t_k = \frac{1}{1-\gamma} < \infty}$$

Infinite number of jumps in a finite time!

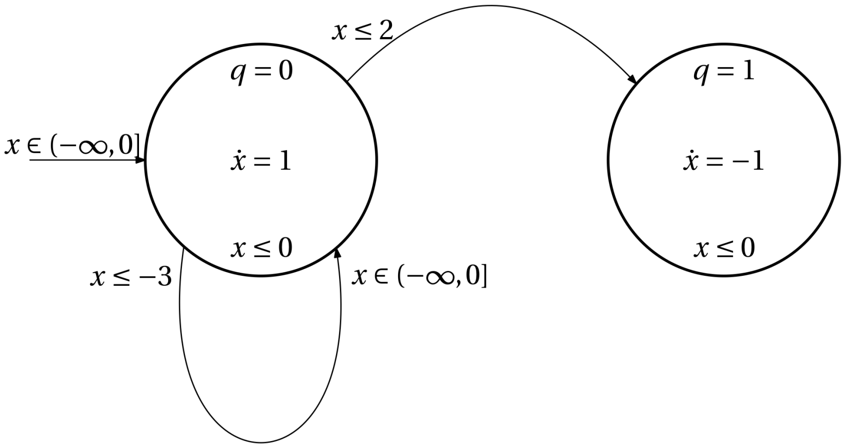
Example: Water tank



$$\max\{Q_{\text{out},2}, Q_{\text{out},2}\} \leq Q_{\text{in}} \leq Q_{\text{out},2} + Q_{\text{out},2}$$



(Non)blocking and (non)determinism



- $x(0) = -3$
- $x(0) = -2$
- $x(0) = -1$
- $x(0) = 0$