Hybrid automata

Models of hybrid systems

Zdeněk Hurák

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Autonomous (no external/control inputs) hybrid automaton

- A tuple of sets and (set) mappings

$$\boxed{\mathcal{H} = \{\mathcal{Q}, \mathcal{Q}_0, \mathcal{X}, \mathcal{X}_0, f, \mathcal{I}, \mathcal{E}, \mathcal{G}, \mathcal{R}\},}$$

where

$\mathcal{Q}$ is a set of discrete states

- Also (operating) modes, also locations.
- Examples:
  - $\mathcal{Q} = \{\text{on}, \text{off}\}$,
  - $\mathcal{Q} = \{\text{working}, \text{broken}, \text{in repair}\}$,
  - $\mathcal{Q} = \{\text{gear}\, 1, \ldots, \text{gear}\, 5\}$.
- It can be characterized either by
  - by enumeration like above, or
  - using a state variable $q(t)$ attaining discrete values.
    - It can also be a vector variable (possibly a binary vector state variable encoding an integer scalar state variable).

$\mathcal{Q}_0 \subseteq \mathcal{Q}$ is a set of initial discrete states

- It can contain a single element.
  - Example: $\mathcal{Q}_0 = \{\mathrm{off}\}$.

$\mathcal{X} \subseteq \mathbb{R}^n$ is a set of continuous states

- Characterized by real-valued vector state variables $\boldsymbol{x}$
  - (often denoted $\boldsymbol{x}(t)$ to emphasize the evolution in time).

$\mathcal{X}_0 \subseteq \mathcal{X}$ is a set of initial continuous states

- A set of values of the (vector) state variable at the initial time.
- Often just a single initial state $\mathcal{X}_0 = \{x_0\}$,
- but it can be useful to set ranges of the values of individual state variables.

$f : \mathcal{Q} \times \mathcal{X} \rightarrow \mathbb{R}^n$ is a vector field parameterized by the location

- Often the dependence on the location q expressed as $f_q(x)$.
- This defines a state equation parameterized by the location:

$$\dot{x}(t) = f_q(x(t)).$$

- It is possible to consider a set-valued map, replacing f with

$$\mathcal{F} : \mathcal{Q} \times \mathcal{X} \rightarrow 2^{\mathbb{R}^n},$$

- which leads to the *differential inclusion*

$$\dot{x} \in \mathcal{F}_q(x).$$

$\mathcal{I}: \mathcal{Q} \rightarrow 2^{\mathcal{X}}$ gives a (location) invariant

- Also called *domain* (of the location).
- Parameterized by the location.
- $\mathcal{I}(q) \subseteq \mathbb{R}^n$.
- A set of values that the state variables is allowed to attain while staying in the given location;
  - if the state of the systems evolves towards the boundary of this set, with a tendency to leave it, the system must be ready to leave that location by transitioning to another location.

$\mathcal{E} \subseteq \mathcal{Q} \times \mathcal{Q}$ is a set of transitions

- The edges of the graph.
- Examples: $\mathcal{E} = \{(\mathrm{off}, \mathrm{on}), (\mathrm{on}, \mathrm{off})\}$.

$\mathcal{G} : \mathcal{E} \rightarrow 2^{\mathcal{X}}$ gives a guard set

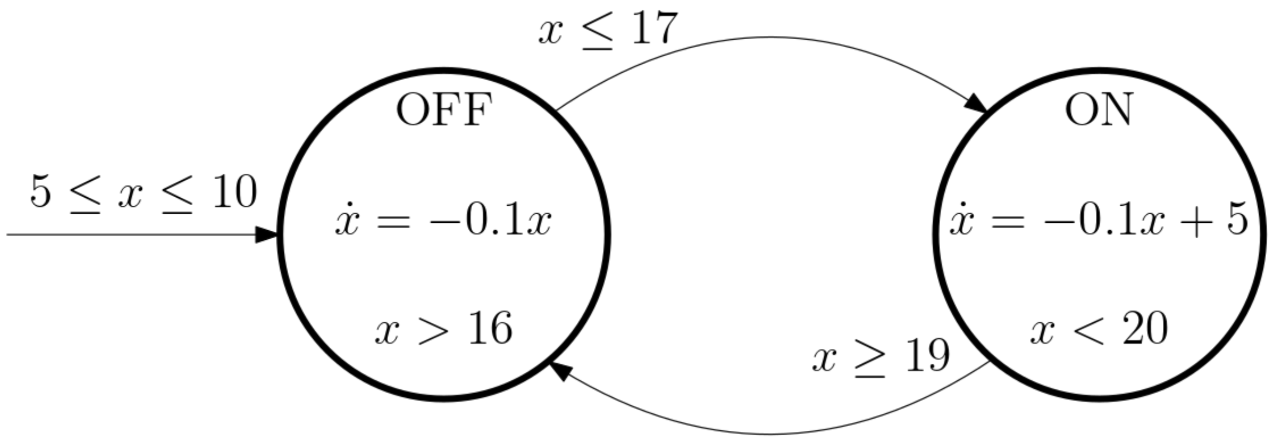
- Associated with the given transition.
- $\mathcal{G}(q_i, q_j)$ is the *guard set* for the transition $(q_i, q_j) \in \mathcal{E}$.
- The *guard condition* for the given transition is satisfied if $x \in \mathcal{G}(q_i, q_j)$.
- If the guard condition is satisfied, the transition is enabled – it may be executed.
- The enabled transition must be executed when the state x leaves the invariant set of the original location.

$\mathcal{R}: \mathcal{E} \times \mathcal{X} \rightarrow 2^{\mathcal{X}}$ is a reset map

- For a given transition from one location to another, it resets the continuous-valued state x to a new value within some subset.
- We also say that the state experiences a *jump*.
- Often the map is single-valued, $r: \mathcal{E} \times \mathcal{X} \rightarrow \mathcal{X}$.
- The state after the jump (associated with the given transitions) is reset according to $x^{+} = r(q_i, q_j, x)$.
- If no resetting of the continuous-valued state takes place, the reset map is just defined as an identity operator with respect to x , that is, $r(q_i, q_j, x) = x$.

Example: Thermostat

- The heater can be turned on or off, which gives two different ODEs for the evolution of temperature in the room.
- The goal is to keep the temperature with at $(18 \pm 2)^\circ$ C.



- Nondeterministic: initial temperatures, switching temperatures.

$$\mathcal{Q} = \{\mathrm{on}, \mathrm{off}\}, \quad \mathcal{Q}_0 = \{\mathrm{off}\}$$

$$\mathcal{X} = \mathbb{R}, \quad \mathcal{X}_0 = \{x : x \in \mathcal{X}, 5 \leq x \leq 10\}$$

$$f_{\mathrm{off}}(x) = -0.1x, \quad f_{\mathrm{on}}(x) = -0.1x + 5$$

$$\mathcal{I}(\mathrm{off}) = \{x \mid x > 16\}, \quad \mathcal{I}(\mathrm{on}) = \{x \mid x < 20\}$$

$$\mathcal{G}(\mathrm{off}, \mathrm{on}) = \{x \mid x \leq 17\}, \ \mathcal{G}(\mathrm{on}, \mathrm{off}) = \{x \mid x \geq 19$$

$\mathcal{R}$ (or r) not specified as the x variable doesn't jump.

Hybrid automaton with external events and control inputs

- Hybrid automaton extended with two new components:
  - a set $\mathcal{A}$ of (external) actions (also events or symbols),
  - a set $\mathcal{U}$ external continuous-valued inputs (control inputs or disturbances).

$$\boxed{\mathcal{H} = \{\mathcal{Q}, \mathcal{Q}_0, \mathcal{X}, \mathcal{X}_0, \mathcal{I}, \mathcal{A}, \mathcal{U}, f, \mathcal{E}, \mathcal{G}, \mathcal{R}\},}$$

where

$\mathcal{A} = \{a_1, a_2, \ldots, a_s\}$ is a set of actions

- The role identical as in a (finite) state automaton: an external event triggers an (enabled) transition from the current discrete state (mode, location) to another.
- Unlike in pure discrete-event systems, here they are within a model that does recognize passing of time – each action must be "time-stamped".

$\mathcal{U} \in \mathbb{R}^m$ is a set of continuous-valued inputs

- Real-valued functions of time.
- Control inputs, disturbances, references, noises.

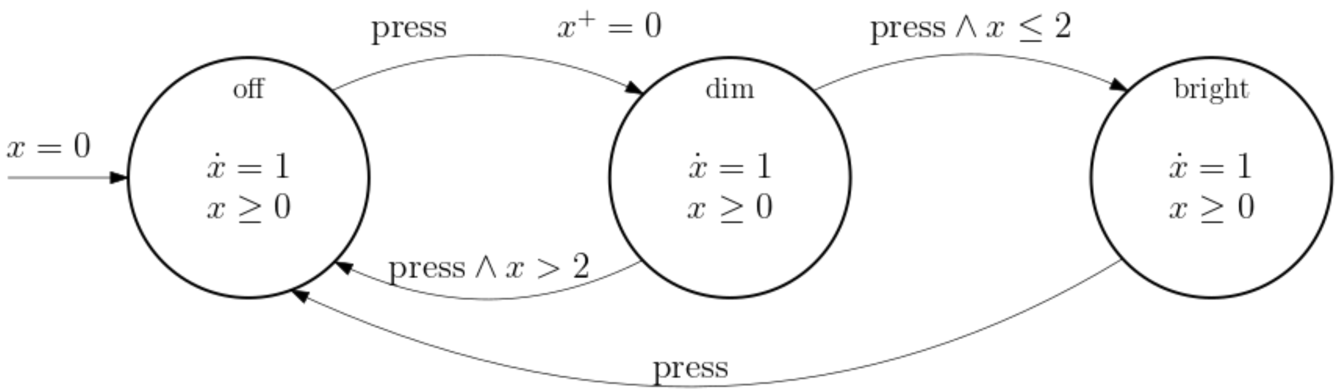
Some modifications needed

upon introduction of these two types of external inputs:

- $f : \mathcal{Q} \times \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^n$ is a vector field that now depends not only on the location but also on the external (control) input, that is, at a given location we consider the state equation $\dot{x} = f_q(x, u)$.
- $\mathcal{E} \subseteq \mathcal{Q} \times (\mathcal{A}) \times \mathcal{Q}$ is a set of transitions now possibly parameterized by the actions (as in classical automata).
- $\mathcal{I} : \mathcal{Q} \rightarrow 2^{\mathcal{X} \times \mathcal{U}}$ is a location invariant now augmented with a subset of the control input set.
- $\mathcal{G} : \mathcal{E} \rightarrow 2^{\mathcal{X} \times U}$ is a guard set now augmented with a subset of the control input set.
- $\mathcal{R} : \mathcal{E} \times \mathcal{X} \times \mathcal{U} \rightarrow 2^{\mathcal{X}}$ is a (state) reset map that is now additionally parameterized by the control input.

- The guard set for the given location gives just necessary (also *enabling*) conditions for the transition ($x \in \mathcal{G}(q)$).
- If enabled, the transition can happen if one of the two things is satisfied:
  - the continous state leaves the invariant set of the given location,
  - an external event occurs.

Example: button-controlled LED



$$\mathcal{Q} = \{\text{off}, \text{dim}, \text{bright}\}, \quad \mathcal{Q}_0 = \{\text{off}\}$$

$$\mathcal{X} = \mathbb{R}, \quad \mathcal{X}_0 = \{0\}$$

$$\mathcal{I}(\text{off}) = \mathcal{I}(\text{bright}) = \mathcal{I}(\text{dim}) = \{x \in \mathbb{R} \mid x \geq 0\}$$

$$f(x) = 1$$

$$\mathcal{A} = \{\text{press}\}$$

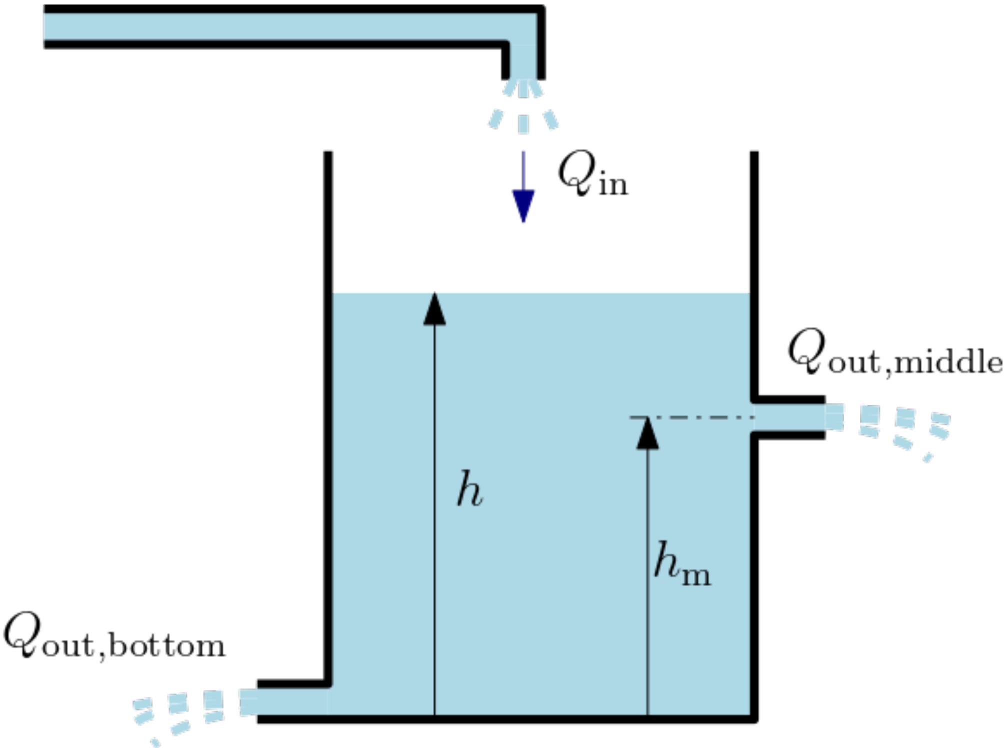
$$\begin{aligned}\mathcal{E} = \{&(\text{off}, \text{press}, \text{dim}), (\text{dim}, \text{press}, \text{off}),\\ &(\text{dim}, \text{press}, \text{bright}), (\text{bright}, \text{press}, \text{off})\}\end{aligned}$$

$$\begin{aligned}\mathcal{G}((\text{off}, \text{press}, \text{dim})) &= \mathcal{X}\\ \mathcal{G}((\text{dim}, \text{press}, \text{off})) &= \{x \in \mathcal{X} \mid x > 2\}\\ \mathcal{G}((\text{dim}, \text{press}, \text{bright})) &= \{x \in \mathcal{X} \mid x \leq 2\}\\ \mathcal{G}((\text{bright}, \text{press}, \text{off})) &= \mathcal{X}.\end{aligned}$$

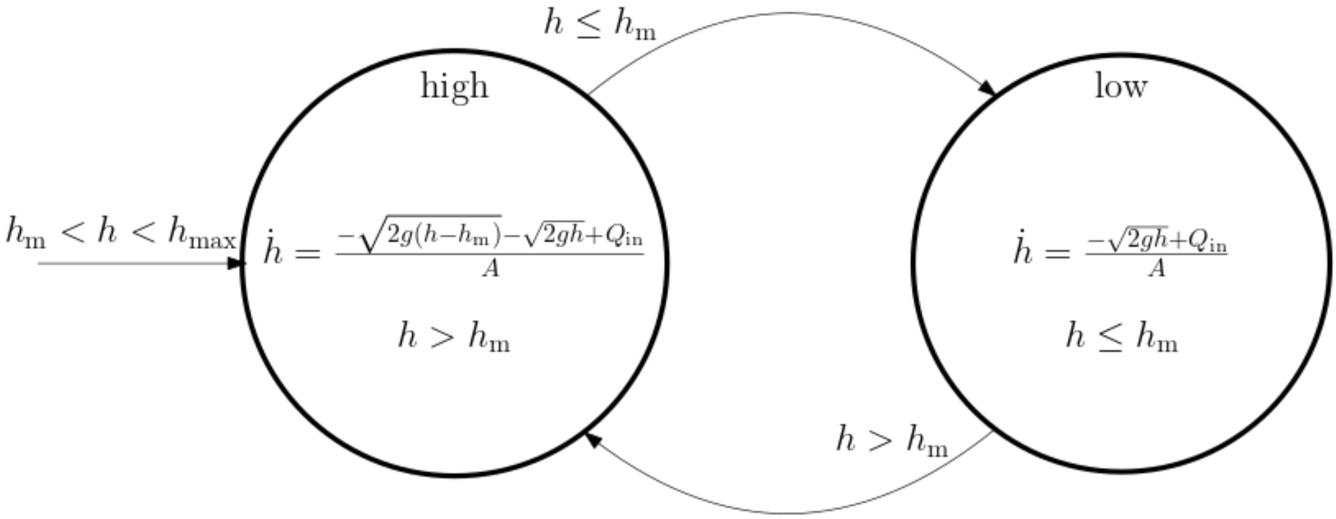
$$r((\mathrm{off}, \mathrm{press}, \mathrm{dim}), x) = 0,$$

- that is, $x^+ = r((\mathrm{off}, \mathrm{press}, \mathrm{dim}), x) = 0$.
- For all other transitions $r((\cdot, \cdot, \cdot), x) = x$,
  - that is, $x^+ = x$.

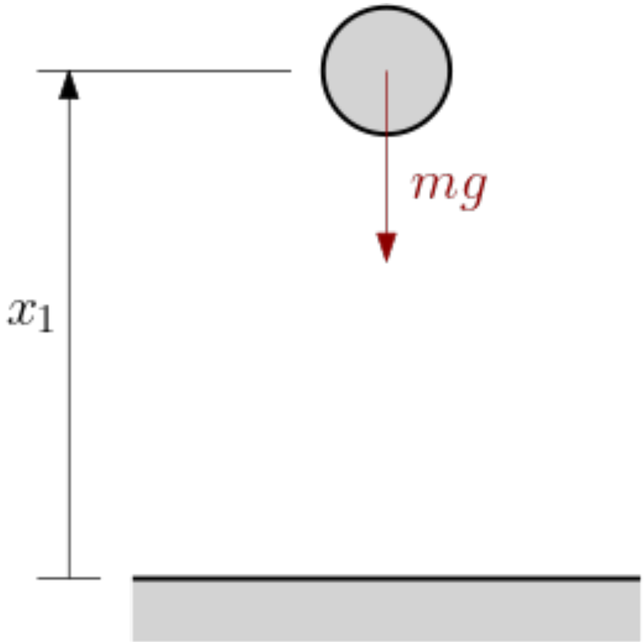
Example: Water tank



$$\dot{V} = \begin{cases} Q_{\text{in}} - Q_{\text{out,middle}} - Q_{\text{out,bottom}}, & h > h_{\text{m}} \\ Q_{\text{in}} - Q_{\text{out,bottom}}, & h \leq h_{\text{m}} \end{cases}$$

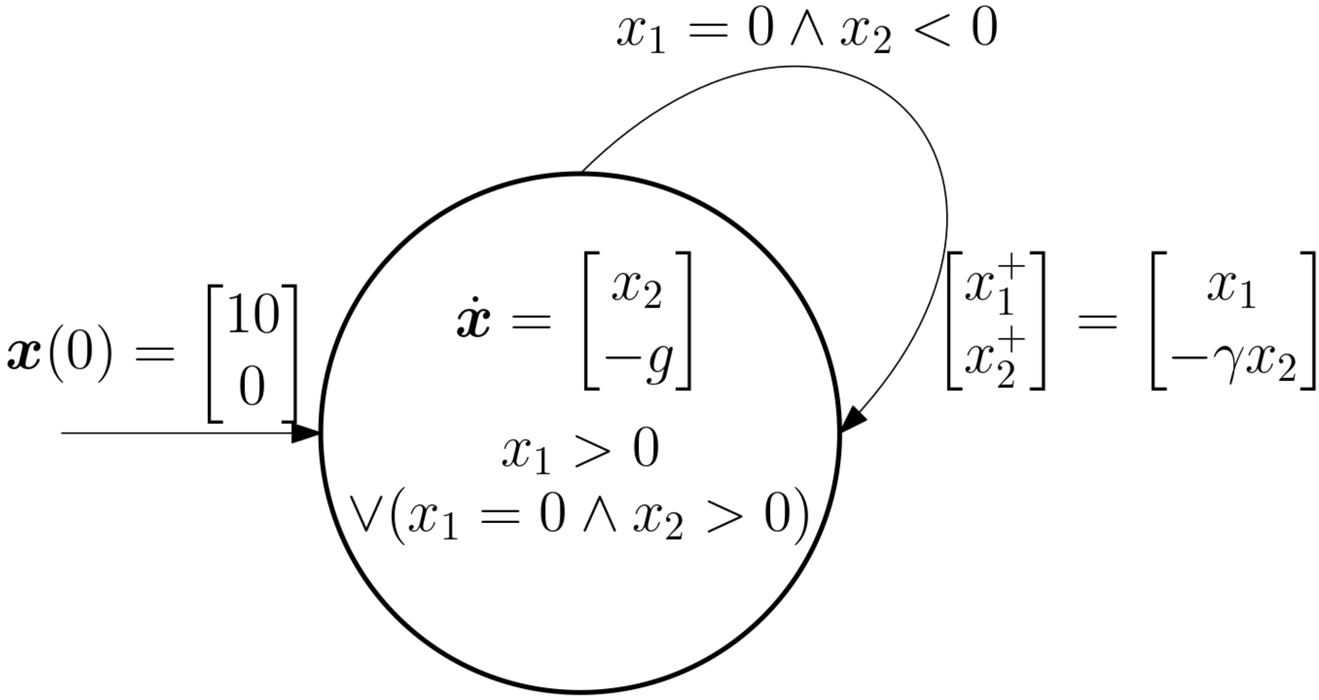


Example: Bouncing ball



$$\dot{\boldsymbol{x}} = \begin{bmatrix} x_2 \\ -g \end{bmatrix}, \quad [10, 0]$$

Example: bouncing ball – HA



$$\mathcal{Q} = \{q\},\ \mathcal{Q}_0 = \{q\}$$

$$\mathcal{X} = \mathbb{R}^2,\ \mathcal{X}_0 = \left\{ \begin{bmatrix} 10 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{I} = \{\mathbb{R}^2 \mid x_1 > 0 \lor (x_1 = 0 \land x_2 > 0)\}$$

$$f(\boldsymbol{x}) = \begin{bmatrix} x_2 \\ -g \end{bmatrix}$$

$$\mathcal{E} = \{(q, q)\}$$

$$\mathcal{G} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 < 0\}$$

$$r((q,q),\boldsymbol{x}) = \begin{bmatrix} x_1 \\ -\gamma x_2 \end{bmatrix},$$

where γ is the coefficient of restitution (e.g., $\gamma = 0.9$).

Comment on the invariant set for the bouncing ball

- Some authors characterize the invariant set as $x_1 \geq 0$.
- But this means that as the ball touches the ground, nothig forces it to leave the location and do the transition.
- Instead, the ball must penetrate the ground, however tiny distance, in order to trigger the transition.
- The present definition avoids this. On the other hand, when solving the model numerically, the use of inequalities, in particular nonstrict ones, is inevitable.

Example: Stick-slip friction (Karnopp)

- Mass m, external force F_a, friction force F_f

$$m\dot v = F_\mathrm{a} - F_\mathrm{f}(v).$$

- Assuming Coulomb friction

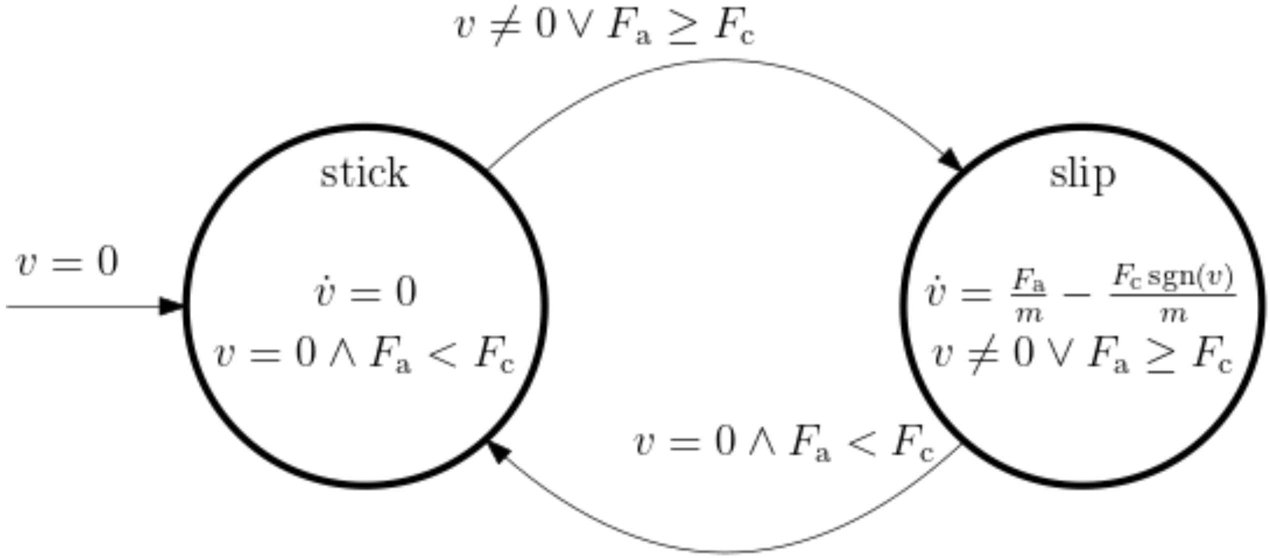
$$F_\mathrm{f}(v) = F_\mathrm{c}\,\mathrm{sgn}(v).$$

- What if $v=0$ and $F_\mathrm{a} < F_\mathrm{c}$?

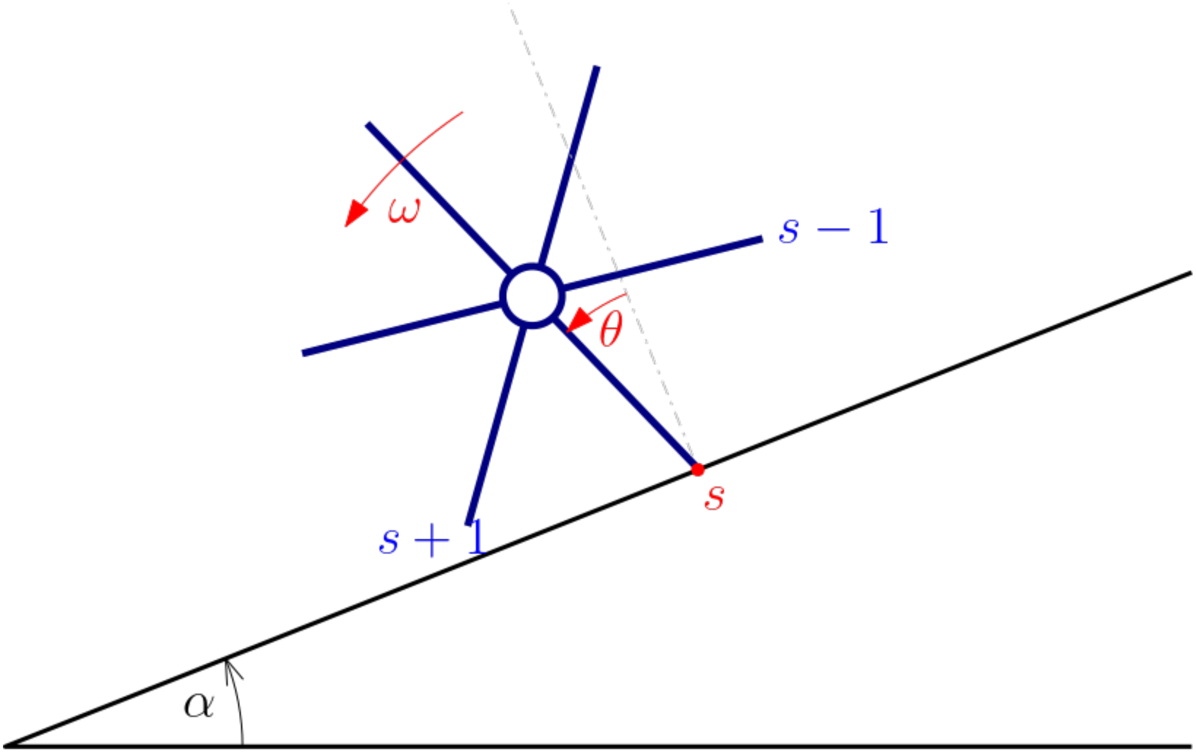
Karnopp model

$$m\dot{v} = 0, \qquad v = 0,\ |F_\mathrm{a}| < F_\mathrm{c}$$

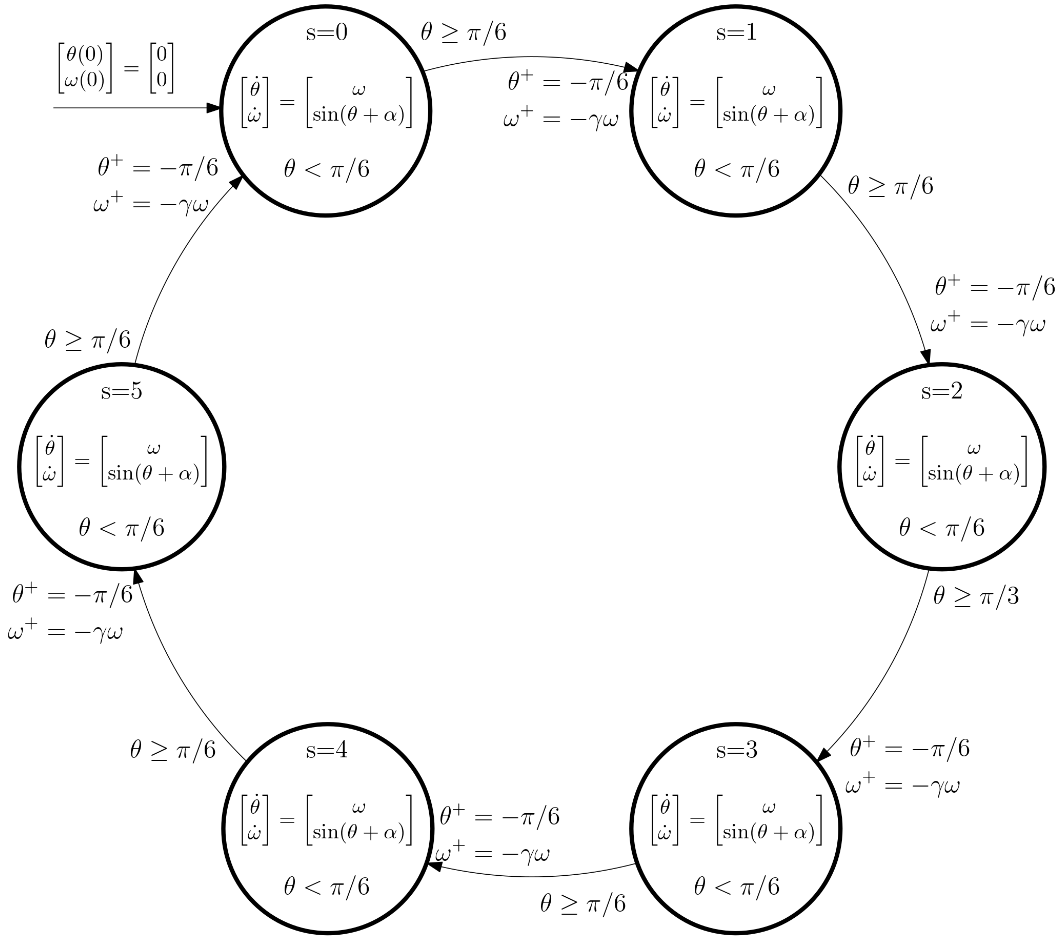
$$F_\mathrm{f} = \begin{cases} \mathrm{sat}(F_\mathrm{a}, F_\mathrm{c}), & v = 0 \\ F_\mathrm{c}\,\mathrm{sgn}(v), & \text{else} \end{cases}$$



Example: Rimless wheel

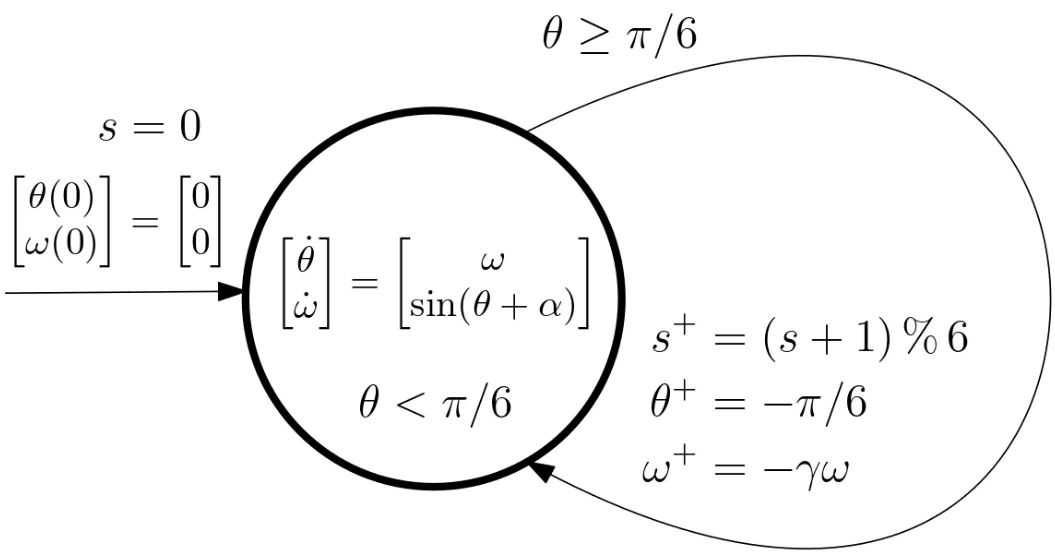


Example: Rimless wheel – HA

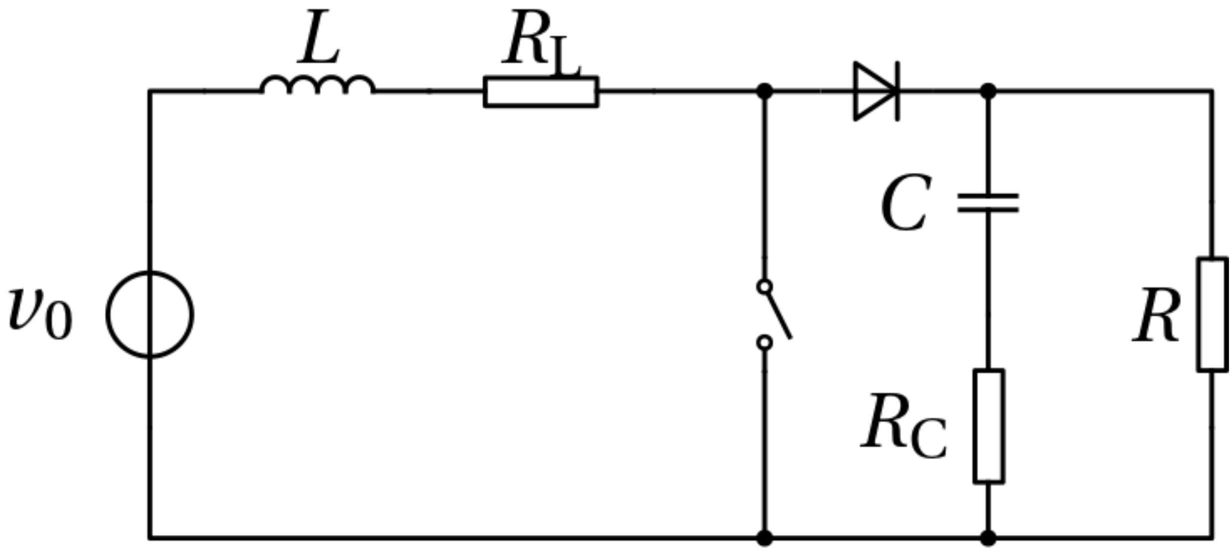


Example: Rimless wheel – alternative HA with a single location

- Do not represent the discrete-state variable graphically as a location (a node in the graph) but rather as another state variable $s \in [0, 1, \ldots, 5]$ within a single location.

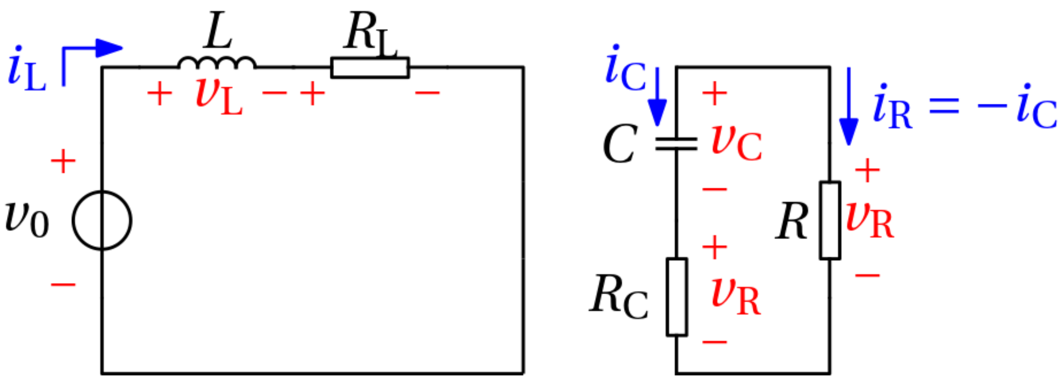


Example: DC-DC boost converter



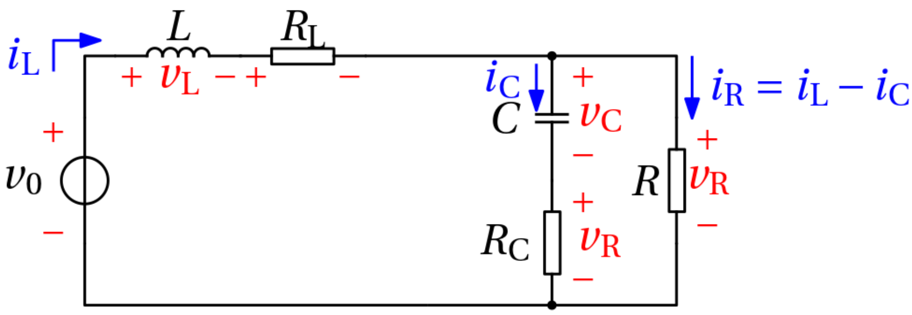
- Not just the switch but also the (ideal) diode is introducing mode transitions.

Example: DC-DC boost converter – the switch closed



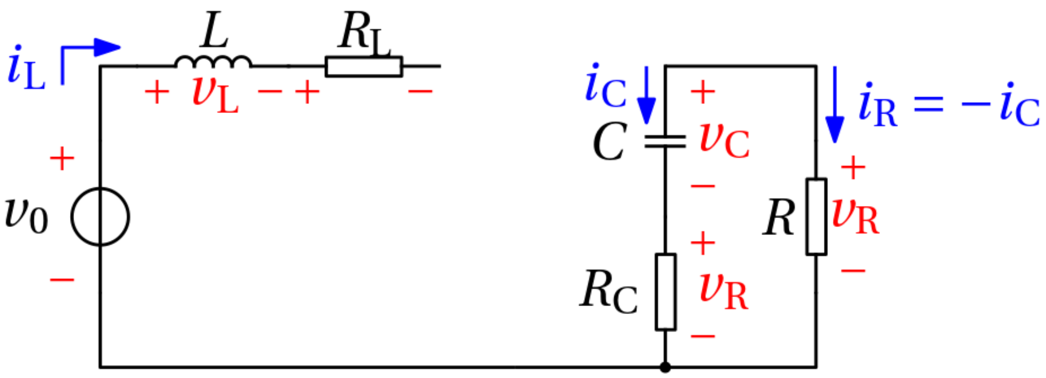
$$\begin{bmatrix} \frac{\mathrm{d}i_L}{\mathrm{d}t} \\ \frac{\mathrm{d}v_C}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\frac{R_L}{L} i_L & 0 \\ 0 & -\frac{1}{C(R+R_C)} \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_0$$

Example: DC-DC boost converter – continuous conduction mode (CCM)



$$\begin{bmatrix} \frac{\mathrm{d}i_L}{\mathrm{d}t} \\ \frac{\mathrm{d}v_C}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\frac{R_L + \frac{RR_C}{R+R_C}}{L} & -\frac{R}{L(R+R_C)} \\ \frac{R}{C(R+R_C)} & -\frac{1}{C(R+R_C)} \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_0$$

Example: DC-DC boost converter – discontinuous cond. mode (DCM)



$$\begin{bmatrix} \frac{\mathrm{d}i_L}{\mathrm{d}t} \\ \frac{\mathrm{d}v_C}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{C(R+R_C)} \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_0$$

Example: DC-DC boost converter – HA