Hybrid (state) equations

as alternative models of hybrid systems

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Also event-flow equations

- The (state) variables can change values discontinuously upon occurence of events – they **jump**,
- between the jumps they evolve continuously – they **flow**.
- Some variables may only flow.
  - Those staying constant between jumps do flow too.
  - But we do not have to distinguish. This is in contrast with hybrid automata, where we do have to distinguish between continuous and discrete variables.

Hybrid (state) equations

$$
\begin{aligned}
\dot x &= f(x), \qquad && x\in\mathcal{C},\\
x^+ &= g(x), \qquad && x\in\mathcal{D}.
\end{aligned}
$$

where

- f is the **flow map**,
- $\mathcal{C}$ is the **flow set**,
- g is the **jump map**,
- $\mathcal{D}$ is the **jump set**.

Hybrid (state) equations – differential inclusions

$$
\begin{aligned}
\dot x &\in \mathcal{F}(x), \qquad & x &\in \mathcal{C},\\
x^+ &\in \mathcal{G}(x), \qquad & x &\in \mathcal{D}.
\end{aligned}
$$

where

- $\mathcal{F}$ is the (set-valued) **flow map**,
- $\mathcal{C}$ is the **flow set**,
- $\mathcal{G}$ is the (set-valued) **jump map**,
- $\mathcal{D}$ is the **jump set**.

Possibly output equations added

$$y(x) = h(x)$$

or

$$y(x) = h(x, u).$$

Example: bouncing ball

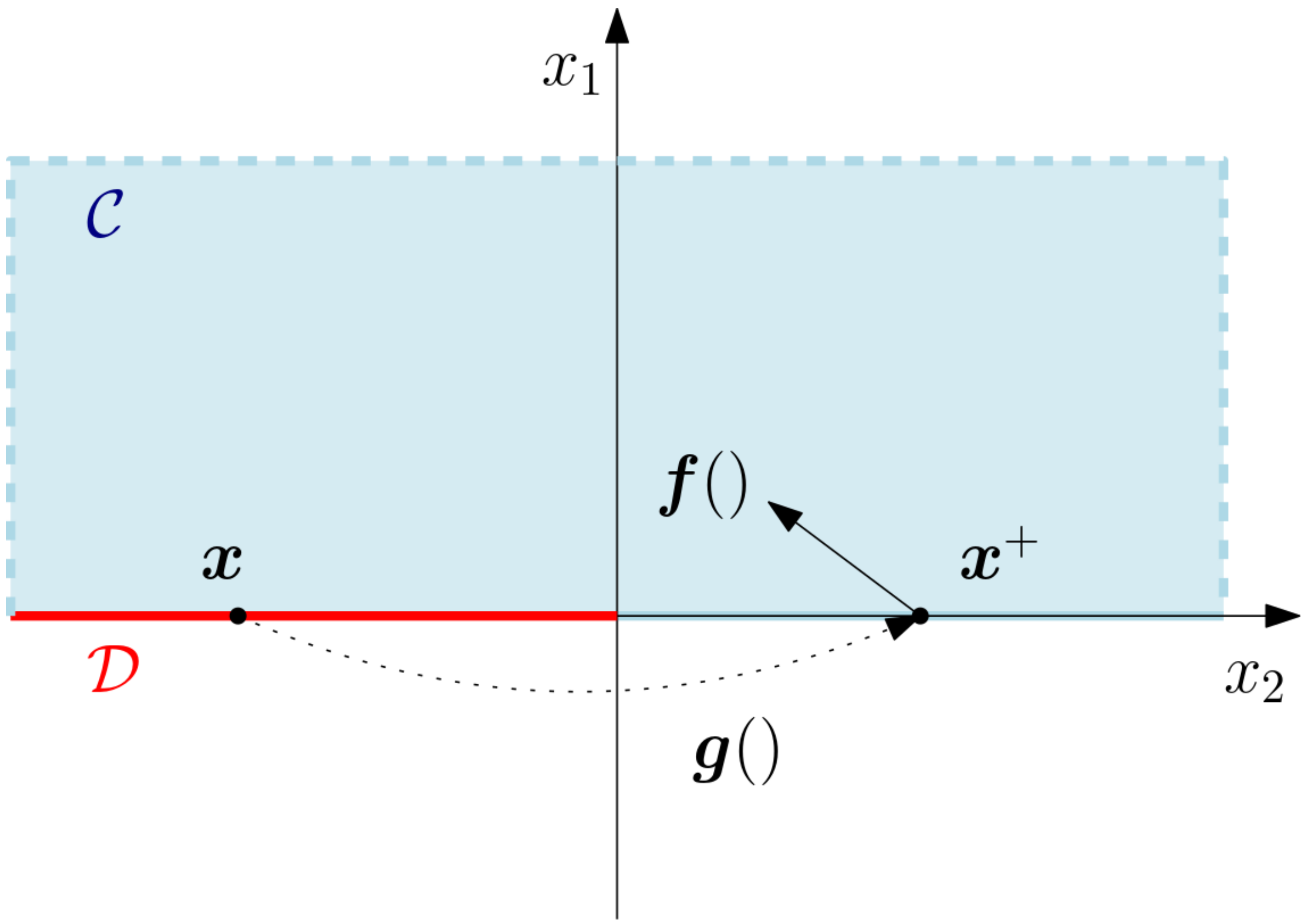
$$\boldsymbol{x} \in \mathbb{R}^2, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 > 0 \vee (x_1 = 0, x_2 \geq 0)\},$$

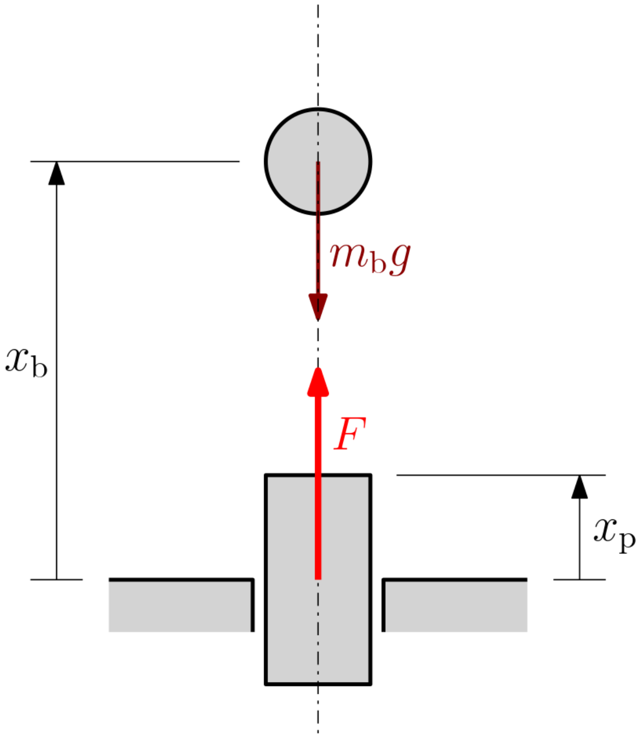
$$f(\boldsymbol{x}) = \begin{bmatrix} x_2 \\ -g \end{bmatrix}, \qquad g = 9.81,$$

$$\mathcal{D} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 = 0, x_2 < 0\},$$

$$g(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ -\alpha x_2 \end{bmatrix}, \qquad \alpha = 0.8.$$


x_1
$\mathcal{C}$
$\boldsymbol{f}()$
$\boldsymbol{x}$
$\boldsymbol{x}^+$
$\mathcal{D}$
x_2
$\boldsymbol{g}()$

Bouncing ball on a controlled piston



- Neglect the sizes (for simplicity).
- The collision happens when $x_b = x_p$,
- and $v_b < v_p$.

- Conservation of momentum after a collision

$$m_b v_b^+ + m_p v_p^+ = m_b v_b + m_p v_p$$

- Collision modelled using a restitution coefficient

$$v_p^+ - v_b^+ = -\gamma(v_p - v_b)$$

- From the momentum conservation

$$v_p^+ = \frac{m_b}{m_p} v_b + v_p - \frac{m_b}{m_p} v_b^+$$

- substitute to the second equation

$$\frac{m_\mathrm{b}}{m_\mathrm{p}}v_\mathrm{b} + v_\mathrm{p} - \frac{m_\mathrm{b}}{m_\mathrm{p}}v_\mathrm{b}^+ - v_\mathrm{b}^+ = -\gamma(v_\mathrm{p} - v_\mathrm{b})$$

- from which we express v_b^+

$$\begin{aligned} v_\mathrm{b}^+ &= \frac{1}{1+\frac{m_\mathrm{b}}{m_\mathrm{p}}}\left(\frac{m_\mathrm{b}}{m_\mathrm{p}}v_\mathrm{b} + v_\mathrm{p} + \gamma(v_\mathrm{p} - v_\mathrm{b})\right)\\ &= \frac{m_\mathrm{p}}{m_\mathrm{p}+m_\mathrm{b}}\left(\frac{m_\mathrm{b}-\gamma m_\mathrm{p}}{m_\mathrm{p}}v_\mathrm{b} + (1+\gamma)v_\mathrm{p}\right)\\ &= \frac{m_\mathrm{b}-\gamma m_\mathrm{p}}{m_\mathrm{b}+m_\mathrm{p}}v_\mathrm{b} + \frac{(1+\gamma)m_\mathrm{p}}{m_\mathrm{p}+m_\mathrm{b}}v_\mathrm{p} \end{aligned}$$

- Substitute to the expression for v_p^+

$$\begin{aligned} v_\mathrm{p}^+ &= \frac{m_\mathrm{b}}{m_\mathrm{p}}v_\mathrm{b} + v_\mathrm{p} - \frac{m_\mathrm{b}}{m_\mathrm{p}}\left(\frac{m_\mathrm{b}-\gamma m_\mathrm{p}}{m_\mathrm{b}+m_\mathrm{p}}v_\mathrm{b} + \frac{(1+\gamma)}{m_\mathrm{p}+}\right.\\ &= \frac{m_\mathrm{b}}{m_\mathrm{p}}\left(1-\frac{m_\mathrm{b}-\gamma m_\mathrm{p}}{m_\mathrm{b}+m_\mathrm{p}}\right)v_\mathrm{b}\\ &\qquad + \left(1-\frac{m_\mathrm{b}}{m_\mathrm{p}}\frac{(1+\gamma)m_\mathrm{p}}{m_\mathrm{p}+m_\mathrm{b}}\right)v_\mathrm{p}\\ &= \frac{m_\mathrm{b}}{m_\mathrm{b}+m_\mathrm{p}}(1+\gamma)v_\mathrm{b} + \frac{m_\mathrm{p}-\gamma m_\mathrm{b}}{m_\mathrm{p}+m_\mathrm{b}}v_\mathrm{p} \end{aligned}$$

- Upon introducing $m = \frac{m_\mathrm{b}}{m_\mathrm{b}+m_\mathrm{p}}$, the jump equation is

$$\begin{bmatrix} v_\mathrm{b}^+ \\ v_\mathrm{p}^+ \end{bmatrix} = \begin{bmatrix} m-\gamma(1-m) & (1+\gamma)(1-m) \\ m(1+\gamma) & 1-m-\gamma m \end{bmatrix}$$

Example: synchronization of fireflies

- n fireflies, x_i is the i-th firefly's clock, normalized to $[0, 1]$.
- The clock resets (to zero) when it reaches 1.
- Each firefly can see the flashing of all other fireflies.
- As soon as it observes a flash, it increases its clock by $\varepsilon\%$.

$$\mathcal{C} = [0,1)^n = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \in [0,1),\ i = 1, \ldots, n\},$$

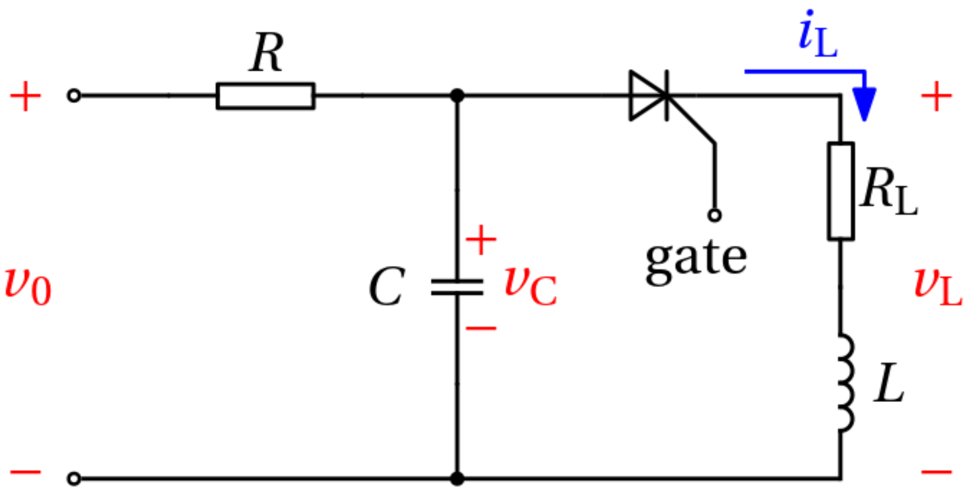
$$\boldsymbol{f} = [f_1, f_2, \ldots, f_n]^\top, \quad f_i = 1, \quad i = 1, \ldots, n,$$

$$\mathcal{D} = \{\boldsymbol{x} \in [0,1]^n \mid \max_i x_i = 1\},$$

$$\boldsymbol{g} = [g_1, \ldots, g_n]^\top,$$

$$g_i(x_i) = \begin{cases} (1+\varepsilon)x_i, & \text{if } (1+\varepsilon)x_i < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example: Thyristor control



Harmonic input voltage generator

$$\begin{aligned}\dot{v}_0 &= \omega v_1\\ \dot{v}_1 &= -\omega v_0\end{aligned}$$

Thyristor *on* (discrete state $q = 1$) or *off* ($q = 0$), firing time τ given by the firing angle $\alpha \in (0, \pi)$.

The state vector

$$\boldsymbol{x} = \begin{bmatrix} u_0\\ u_1\\ i_L\\ v_C\\ q\\ \tau \end{bmatrix}$$

The flow map

$$
\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \omega v_1 \\ -\omega v_0 \\ q\frac{v_\mathrm{C}-Ri_\mathrm{L}}{L} \\ -\frac{1}{CR}v_\mathrm{C} + \frac{1}{CR}v_0 - \frac{1}{C}i_\mathrm{L} \\ 0 \\ 1 \end{bmatrix}
$$

The flow set

$$
\begin{aligned}
\mathcal{C} = \{\boldsymbol{x} \mid q=0,\, \tau < \frac{\alpha}{\omega},\, i_\mathrm{L} = 0\}\\
\cup \{\boldsymbol{x} \mid q=1,\, i_\mathrm{L} > 0\}
\end{aligned}
$$

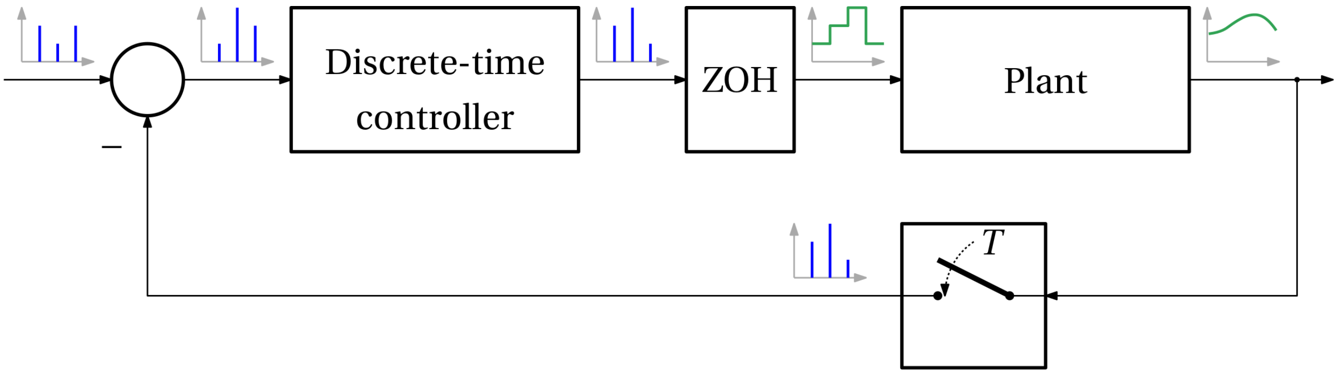
The jump set

$$
\begin{aligned}
\mathcal{D} = \{\boldsymbol{x} \mid q=0,\, \tau \geq \frac{\alpha}{\omega},\, i_\mathrm{L} = 0,\, v_\mathrm{C} > 0\}\\
\cup \{\boldsymbol{x} \mid q=1,\, i_\mathrm{L} = 0,\, v_\mathrm{C} < 0\}
\end{aligned}
$$

The jump map

$$\begin{bmatrix} u_0 \\ u_1 \\ \vdots \end{bmatrix}$$

Example: Sampled-data feedback



- Plant modelled by $\dot{x}_\mathrm{p} = f_\mathrm{p}(x_\mathrm{p}, u),\; y = h(x_\mathrm{p})$.
- Controller samples the output T-periodically and computes its own output as a nonlinear function $u = \kappa(r - y)$.
- The closed-loop model is then

$$\dot{x}_\mathrm{p} = f_\mathrm{p}(x_\mathrm{p}, \kappa(r - h(x_\mathrm{p}))),\; y = h(x_\mathrm{p}).$$

- The closed-loop state vector

$$\boldsymbol{x} = \begin{bmatrix} x_\mathrm{p} \\ u \\ \tau \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$$

- The flow set

$$\mathcal{C} = \{\boldsymbol{x} \mid \tau \in [0, T)\}$$

- The flow map

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_\mathrm{p}(x_\mathrm{p}, u)\\ 0\\ 1 \end{bmatrix}$$

- The jump set

$$\mathcal{D} = \{\boldsymbol{x} \mid \tau = T\}$$

- or rather

$$\mathcal{D} = \{\boldsymbol{x} \mid \tau \geq T\}$$

- The jump map

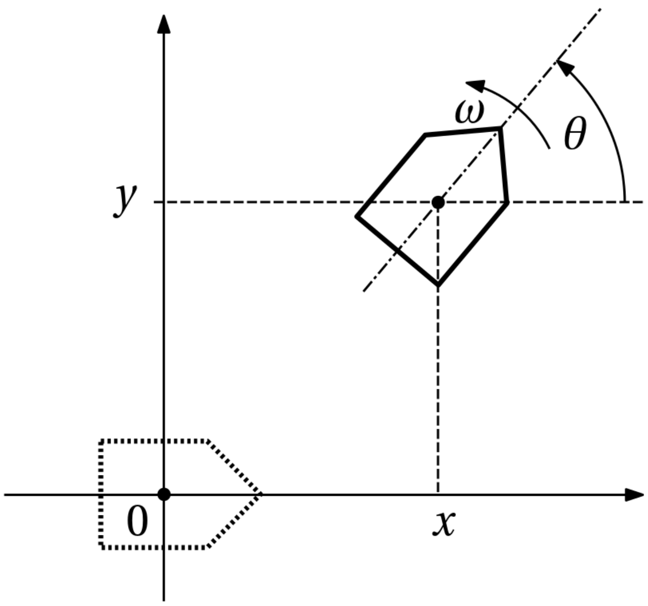
$$\boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} x_\mathrm{p}\\ \kappa(r-y)\\ 0 \end{bmatrix}$$

Hybrid what, when closing the loop?

- Hybrid plant + continuous controller.
- Hybrid plant + hybrid controller.
- Continuous plant + hybrid controller.

Impossibility to stabilize without a hybrid controller

Example: unicycle stabilization



$$
\begin{aligned}
\dot x &= v\cos\theta\\
\dot y &= v\sin\theta\\
\dot\theta &= \omega
\end{aligned}
$$

$$
\boldsymbol{x} = \begin{bmatrix}x\\ y\\ \theta\end{bmatrix},\quad \boldsymbol{u} = \begin{bmatrix}v\\ \omega\end{bmatrix}
$$

Condition of stabilizability by a time-invariant continuous state feedback

- Image of every neighborhood of the origin under $(\boldsymbol{x},\boldsymbol{u}) \mapsto \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$ contains some neighborhood of the origin.
    - Brockett, R. 'Asymptotic Stability and Feedback Stabilization'. In Differential Geometric Control Theory, edited by R. Brockett, R. Millman, and H. Sussmann. Boston: Birkhäuser, 1983.

In the example

$$\begin{bmatrix} x \\ y \\ \theta \\ v \\ \omega \end{bmatrix} \mapsto \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$$

- Now consider a neighborhood such that $|\theta| < \frac{\pi}{2}$.
- It is impossible to get $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) = \begin{bmatrix} 0 \\ f_2 \\ 0 \end{bmatrix},\; f_2 \neq 0$.
- Hence, stabilization by a continuous feedback $\boldsymbol{u} = \kappa(\boldsymbol{x})$ is impossible.

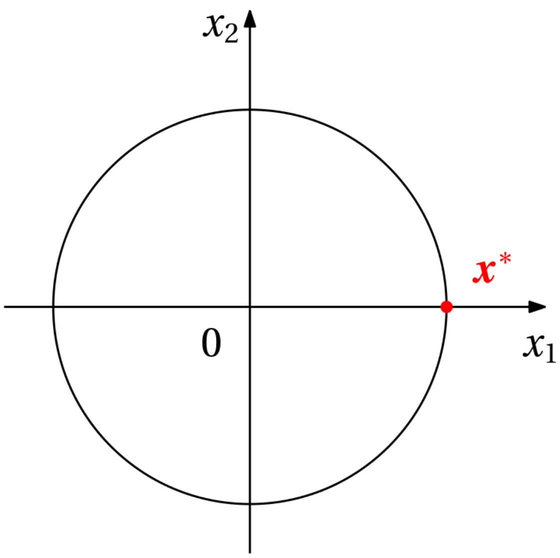
Ex.: global stabilization on a circle

- Demonstration of a more general phenomenon of stabilization on a manifold.
- Even if stabilization by a continous feedback is possible, it might not guarantee global stability.
- The dynamics of a particle sliding around a unit circle

$$\dot{\boldsymbol{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{x},$$

where $\boldsymbol{x} \in \mathbb{S}^1, \quad u \in \mathbb{R}.$

- The point to be stabilized is $\boldsymbol{x}^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



- What is required from a globally stabilizing controller?
  - Solutions stay in $\mathbb{S}^1$,
  - Solutions converge to $\boldsymbol{z}^*$,
  - If a solution starts near $\boldsymbol{z}^*$, it stays near.
- One candidate is $\kappa(\boldsymbol{x}) = -x_2$.
- Define the (Lyapunov) function $V(\boldsymbol{x}) = 1 - x_1$.
- Its time derivative along the solution trajectory

$$
\begin{aligned}
\dot{V} &= \left(\nabla_{\boldsymbol{x}} V\right)^\top \dot{\boldsymbol{x}}\\
&= \begin{bmatrix}-1 & 0\end{bmatrix} \left(-x_2 \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)\\
&= -x_2^2\\
&= -(1 - x_1^2).
\end{aligned}
$$

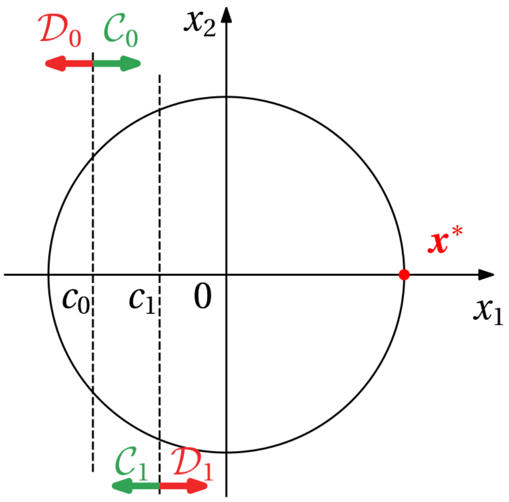
- It follows that

$$
\dot{V} < 0 \quad \forall \boldsymbol{x} \in \mathbb{S}^1 \setminus \{-1, 1\}.
$$

- With $u = -x_2$ the point $\boldsymbol{x}^*$ is stable but not globally atractive.

Example continued: Hybrid control

- Introduce new state variable $q \in \{0, 1\}$ (to get hysteresis).



$$\kappa(\boldsymbol{x}, 0) = \kappa_0(\boldsymbol{x}) = -x_2$$
$$\kappa(\boldsymbol{x}, 1) = \kappa_1(\boldsymbol{x}) = -x_1$$

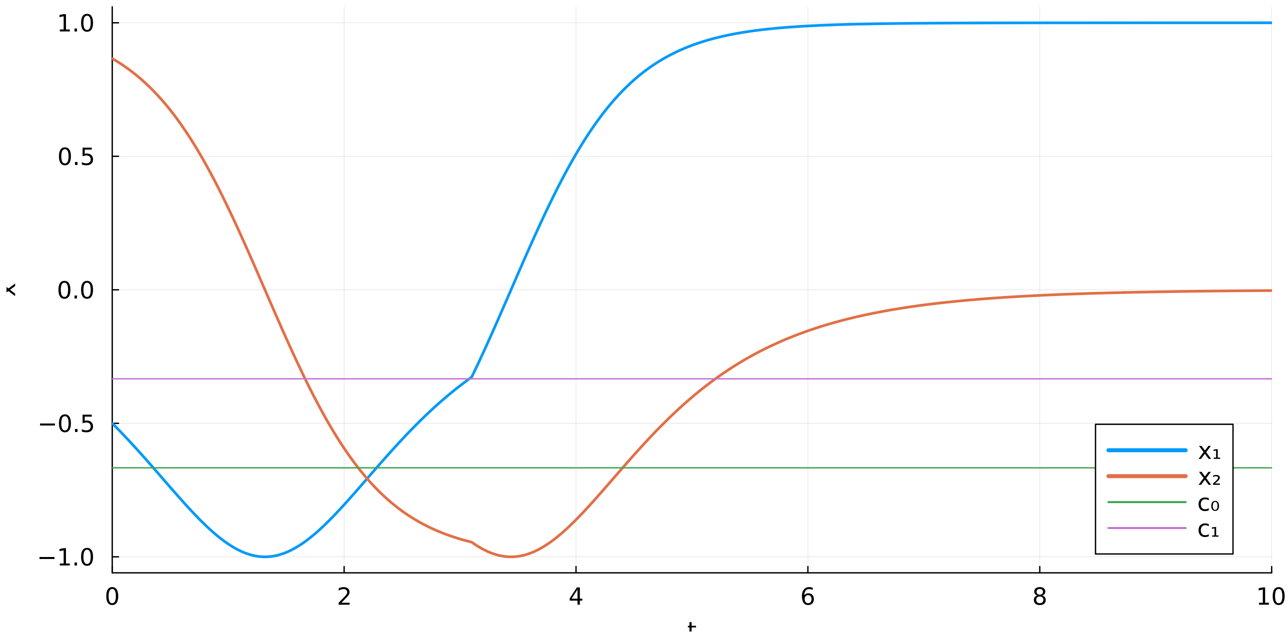
$$\mathcal{C} = (\mathcal{C}_0 \times \{0\}) \cup (\mathcal{C}_1 \times \{1\})$$

$$f(\boldsymbol{x}, q) = \ldots$$

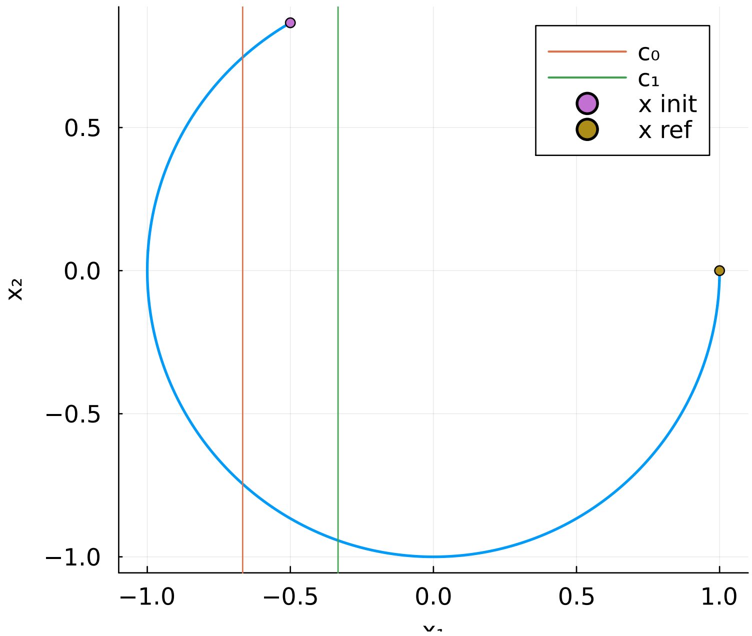
$$\mathcal{D} = (\mathcal{D}_0 \times \{0\}) \cup (\mathcal{D}_1 \times \{1\})$$

$$g(\boldsymbol{x}, q) = 1 - q \quad \forall [\boldsymbol{x}, q]^\top \in \mathcal{D}$$

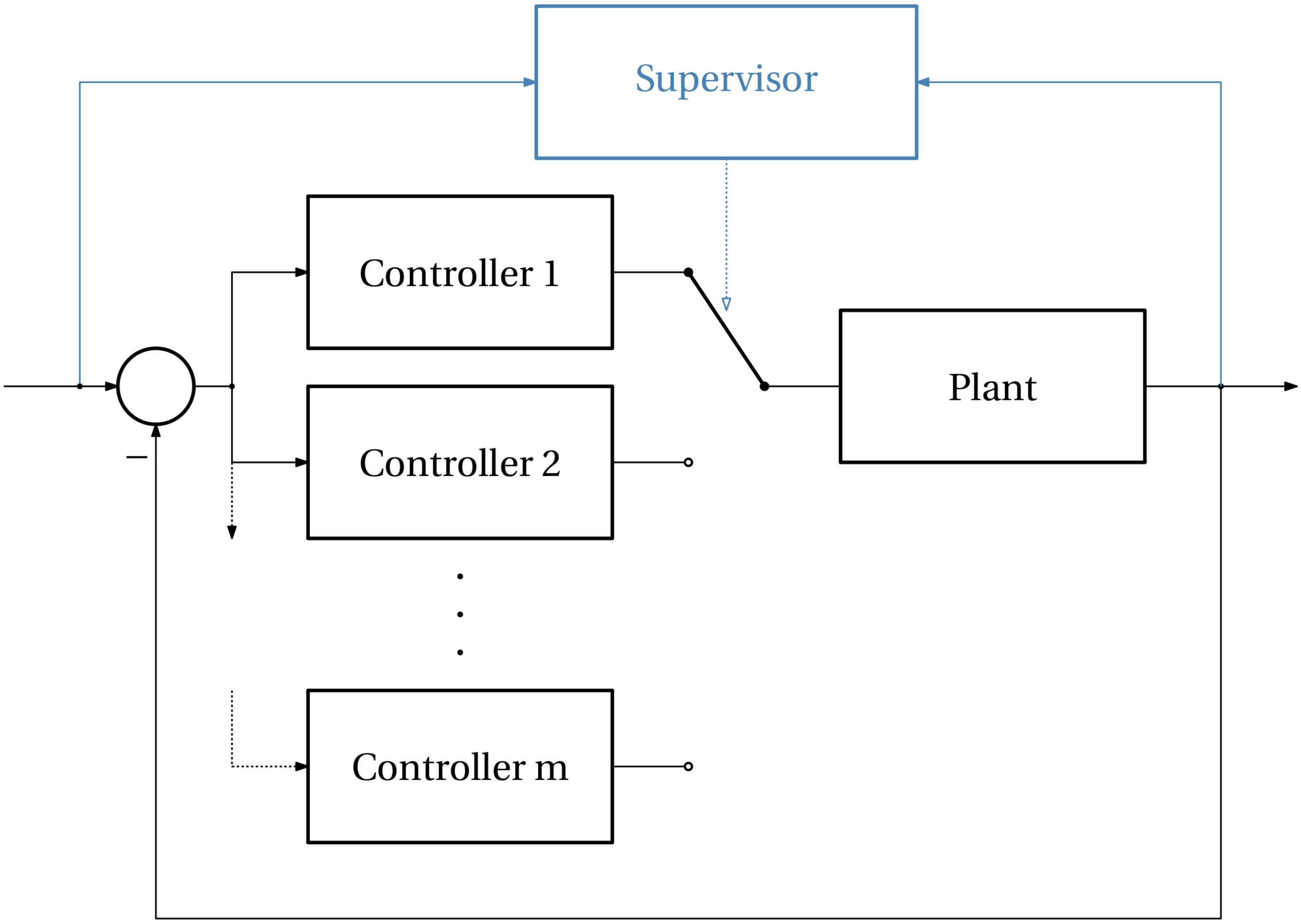
Example continued



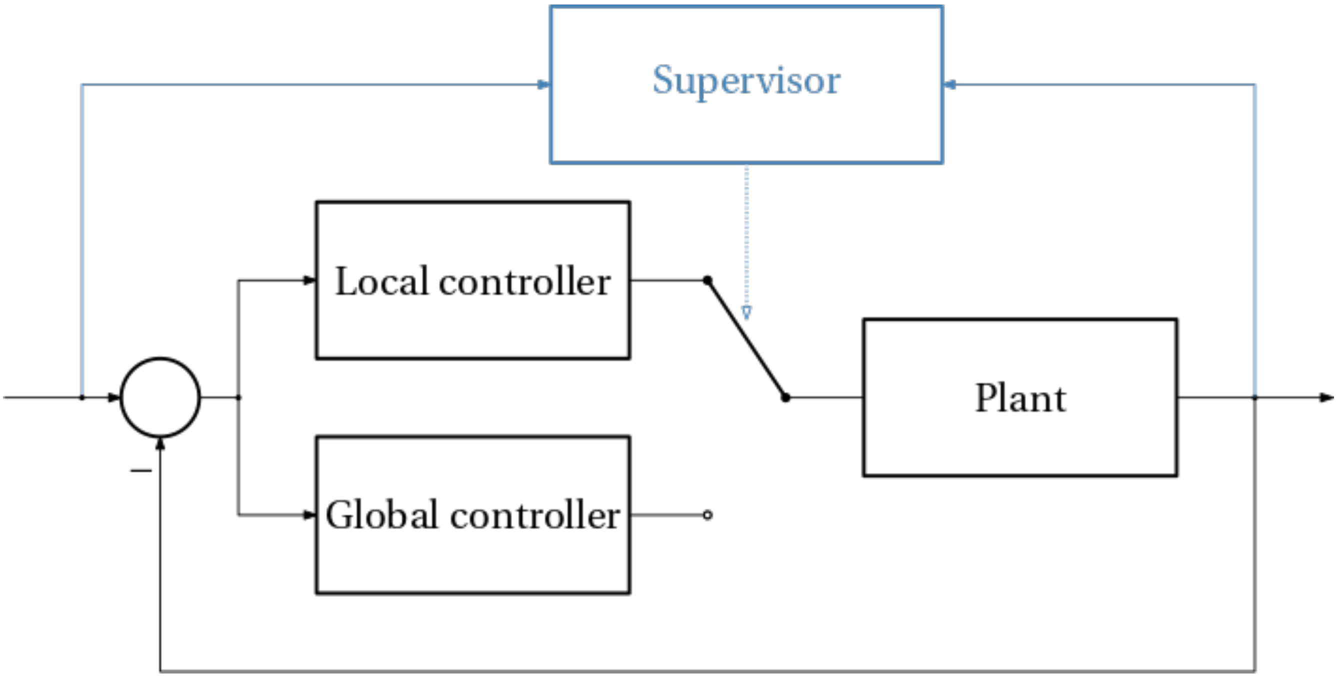
Example continued



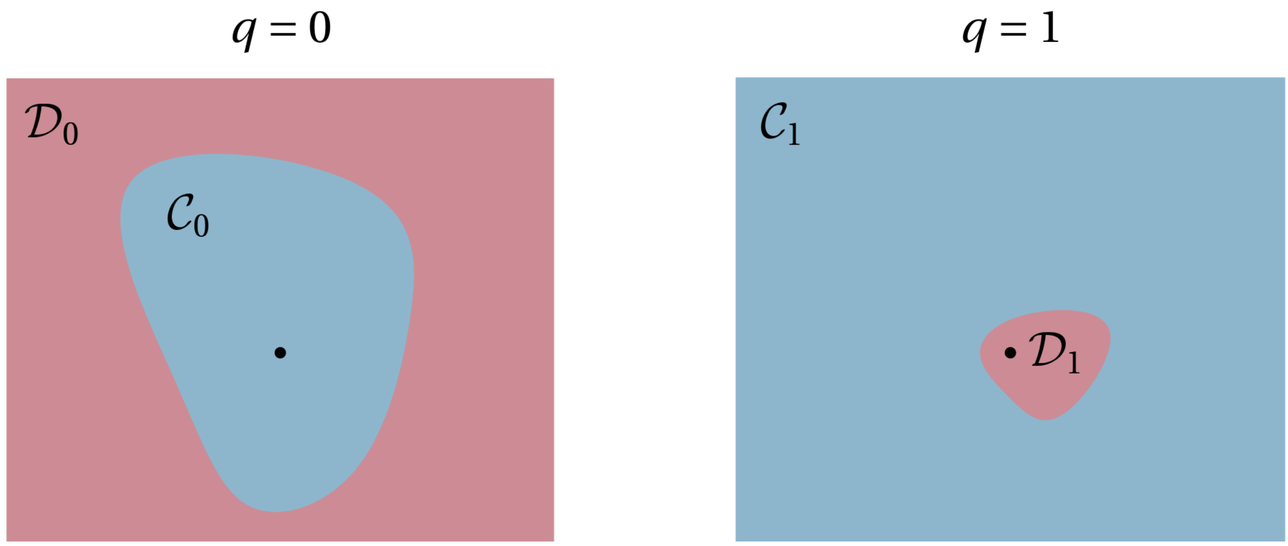
Supervisory control



Combining local and global controllers ($\subset$ supervisory control)



- Local controllers have good transient response but only work well in a small region around the equilibrium state.
- Global controllers have poor transient response but work well in a larger region around the equilibrium state.
- For example: pendulum swingup and stabilization.
- The flow and jump sets for the local and global controllers (which is which?)



Software tools

- Sanfelice, Ricardo, and Pablo Nanez. 'Hybrid Equations (HyEQ) Toolbox for Matlab'. Version 3.0, 15 October 2022. https://github.com/pnanez/HyEQ_Toolbox. Documentation at https://hyeq.github.io.

Literature

- Goebel, R., R. G. Sanfelice, and A. R. Teel. ‘Hybrid Dynamical Systems’. IEEE Control Systems Magazine 29, no. 2 (April 2009): 28–93. https://doi.org/10.1109/MCS.2008.931718.
- Goebel, Rafal, Ricardo G. Sanfelice, and Andrew R. Teel. Hybrid Dynamical Systems: Modeling, Stability, and Robustness. Princeton University Press, 2012.
- Sanfelice, Ricardo G. Hybrid Feedback Control. Princeton University Press, 2021.