(Some) classes of hybrid systems

Reset, switched, and piecewise affine systems

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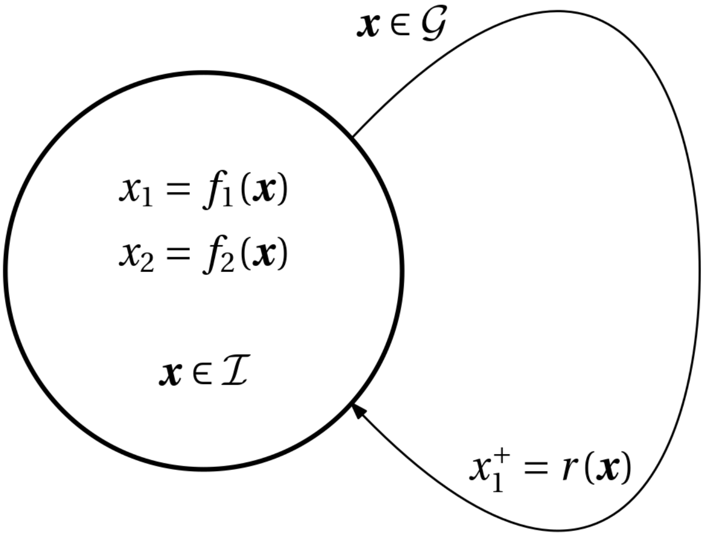
Hybrid systems (B3M35HYS) Fall 2023

Reset (also impulsive) systems

- Some variables reset (jump) and flow, some only flow,
- but no variables that only reset (no discrete states).



- The bouncing ball is one example,
- another one is ->

Example: Reset oscillator

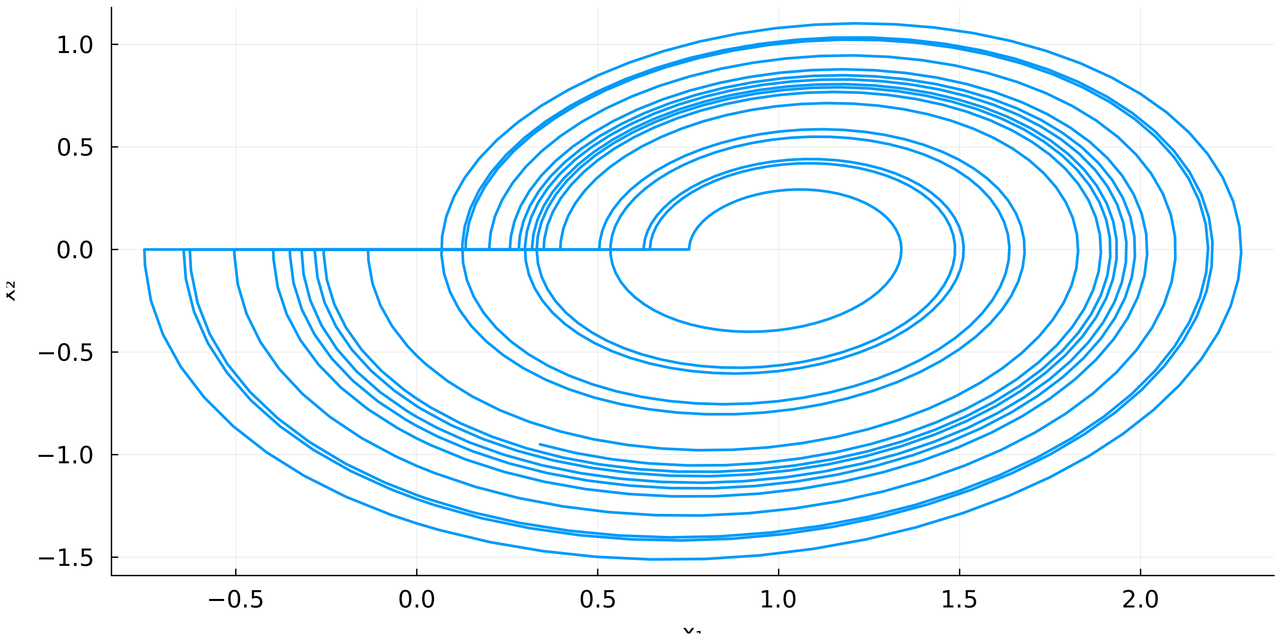
$$\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 2\delta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}, \quad \boldsymbol{x}\in\mathcal{C},$$

$$x_1^+ = -x_1, \quad \boldsymbol{x}\in\mathcal{D},$$

where

$$\begin{aligned}\mathcal{D} &= \{\boldsymbol{x}\in\mathbb{R}^2 \mid x_1 < 0, x_2 = 0\}\\ \mathcal{C} &= \mathbb{R}^2 \setminus \mathcal{D}.\end{aligned}$$

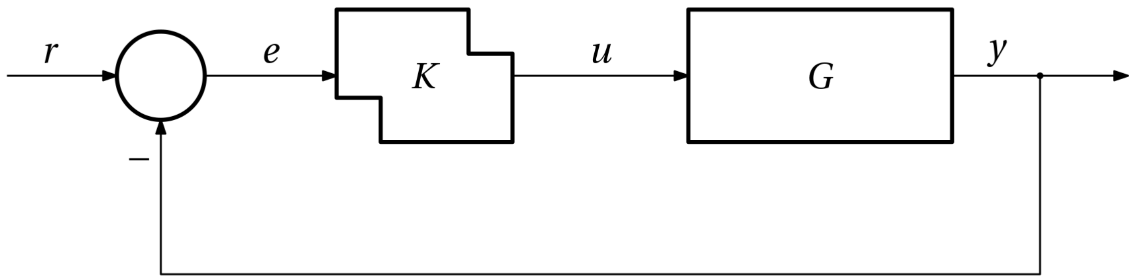
Reset oscillator simulation outcomes



Clegg's integrator (CI)

- As soon as the sign of the input changes, the integrator resets to zero.
  - The integrator that the sign of its input and output identical.
- Originally presented in the form of an analog circuit (opamps, diodes, resistors, capacitors).
- Unlike the traditional (linear) integrator, the CI exhibits much smaller phase lag (some 38 vs 90 deg).

First-order reset element (FORE)



- plant $G(s) = \frac{s+1}{s(s+0.2)}$
- controller

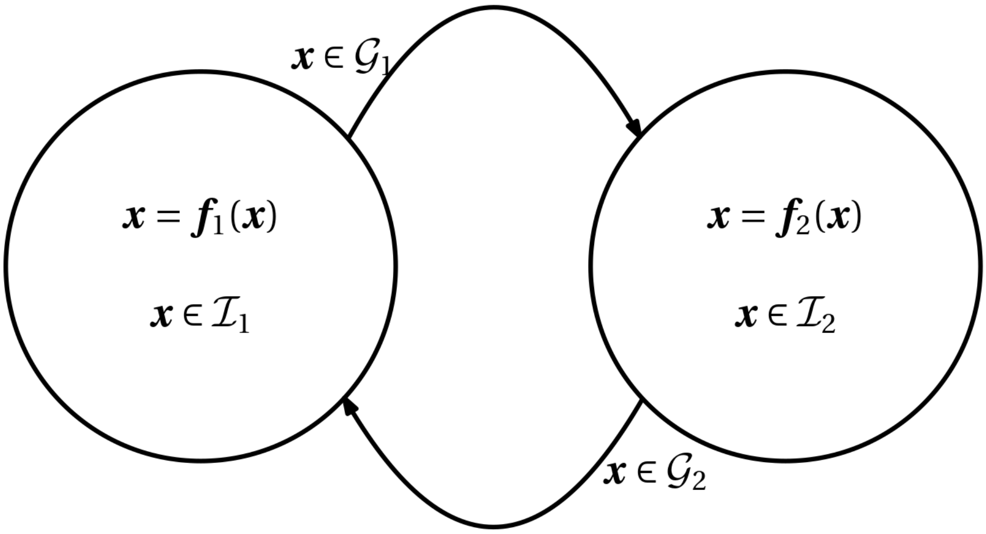
$$
\begin{aligned}
\dot u &= au + ke, &\quad \text{when } e \neq 0,\\
u^+ &= 0, &\quad \text{when } e = 0.
\end{aligned}
$$

When (not) to use reset control?

- Cannot another linear controller be found that performs even bettern than the FORE?
- Cannot it happen that upon introducing resetting into a linear controller, closed-loop stability is lost?
- Use reset control with care.
- Can be helfpul if the linearly-controller plant is subject to *fundamental limitations of achievable control performance*:
  - integrators and unstable poles,
  - zeros in the right half-plane (non-minimum phase),
  - delays,
  - …

Switched systems

- Some variables reset (reset) and stay constant between resets (they model discrete states),
- and some variables only flow and do not reset.



Switched systems

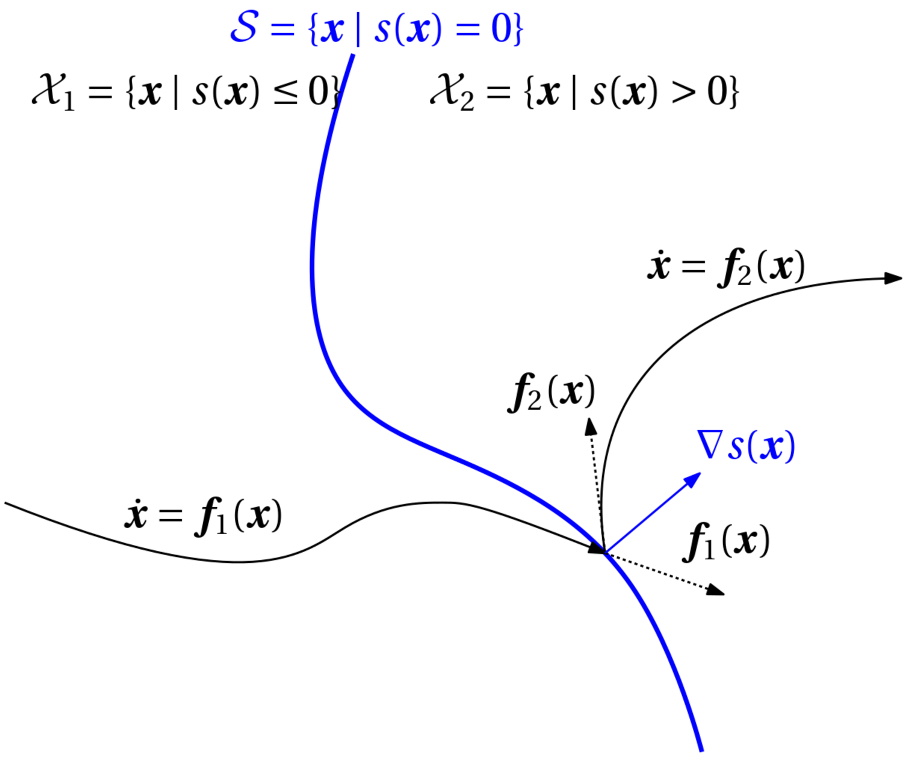
$$\dot{\boldsymbol{x}} = \boldsymbol{f}_q(\boldsymbol{x})$$

$q \in \{1, 2, \ldots, m\}$ can be

- time-driven: $\dot{\boldsymbol{x}} = \boldsymbol{f}_{q(t)}(\boldsymbol{x})$
- state-driven:

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{f}_1(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \mathcal{X}_1, \\ \vdots \\ \boldsymbol{f}_m(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \mathcal{X}_m. \end{cases}$$

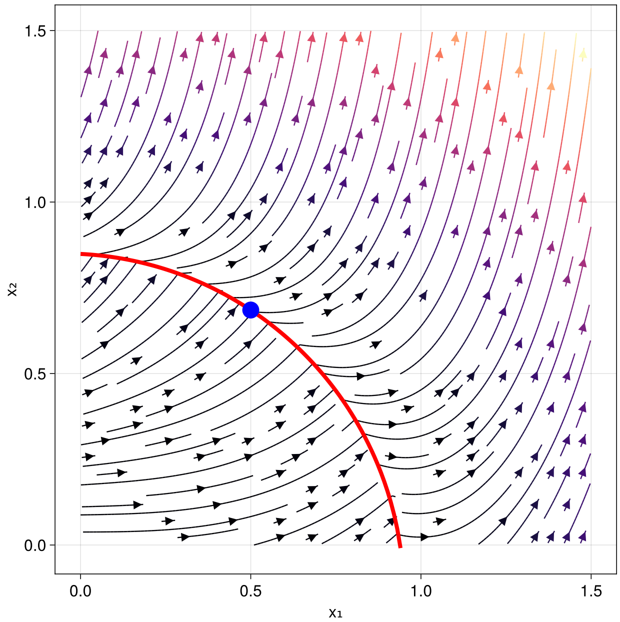
State-dependent switching



Example: The flow transverses the boundary

$$\boldsymbol{f}_1(\boldsymbol{x}) = \begin{bmatrix} 1 \\ x_1^2 + 2x_2^2 \end{bmatrix}$$

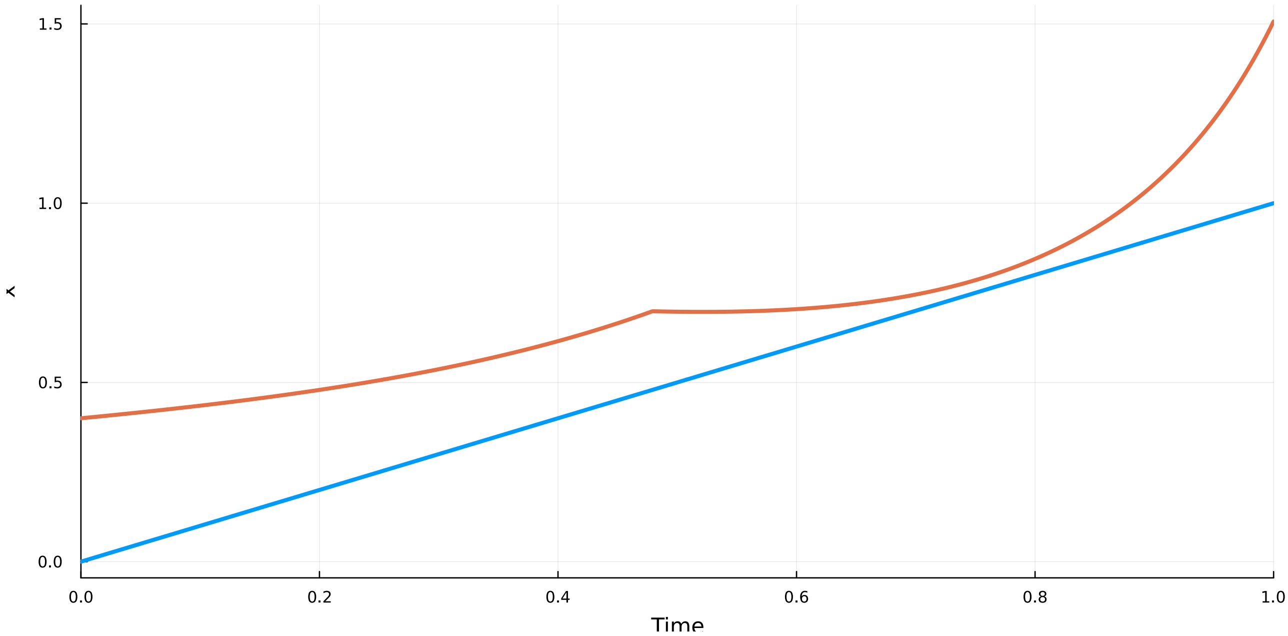
$$\boldsymbol{f}_2(\boldsymbol{x}) = \begin{bmatrix} 1 \\ 2x_1^2 + 3x_2^2 - 2 \end{bmatrix}$$



$$s(x_1, x_2) = (x_1 + 0.05)^2 + (x_2 + 0.15)^2 - 1$$

$$(\nabla s)^\top \boldsymbol{f}_1\Big|_{\boldsymbol{x}_0} \geq 0, \quad (\nabla s)^\top \boldsymbol{f}_2\Big|_{\boldsymbol{x}_0} \geq 0$$

Solution?

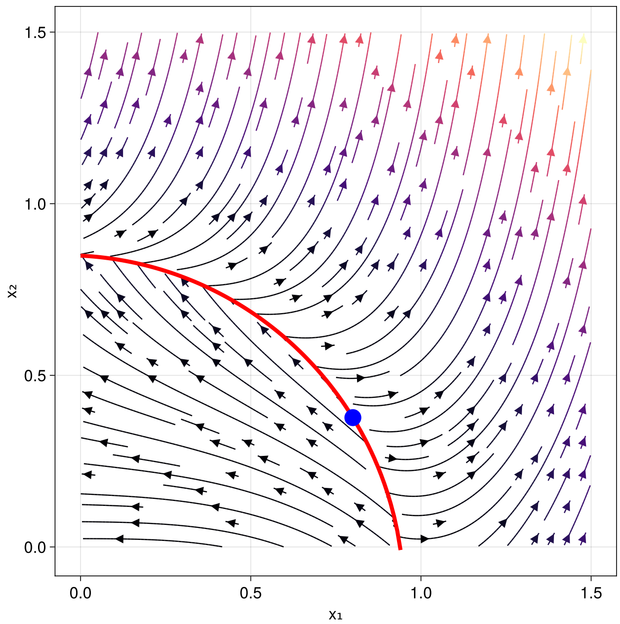


- This solution does not satisfy the differential equation on the boundary of the two domains (the derivative of x_2 does not exist there).

Example: The flow pulls away from the boundary

$$\boldsymbol{f}_1(\boldsymbol{x}) = \begin{bmatrix} -1 \\ x_1^2 + 2x_2^2 \end{bmatrix}$$

$$\boldsymbol{f}_2(\boldsymbol{x}) = \begin{bmatrix} 1 \\ 2x_1^2 + 3x_2^2 - 2 \end{bmatrix}$$

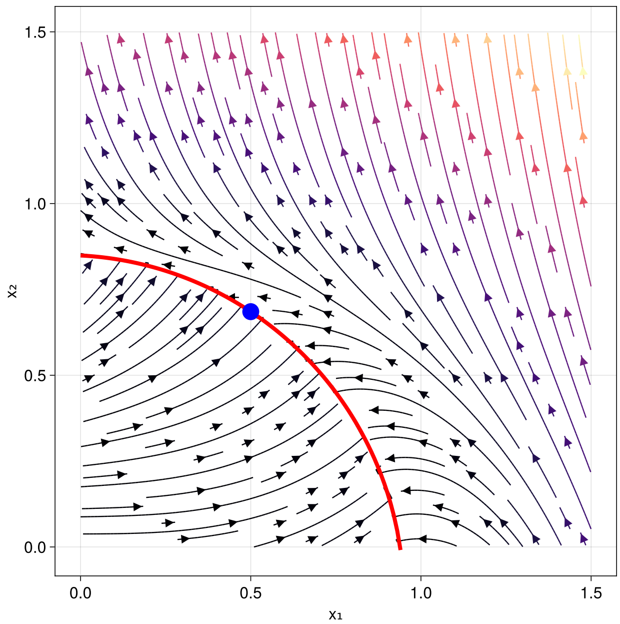


$$(\nabla s)^\top \boldsymbol{f}_1\Big|_{\boldsymbol{x}_0} \leq 0, \quad (\nabla s)^\top \boldsymbol{f}_2\Big|_{\boldsymbol{x}_0} \geq 0$$

Example: The flow pushes towards the boundary

$$\boldsymbol{f}_1(\boldsymbol{x}) = \begin{bmatrix} 1 \\ x_1^2 + 2x_2^2 \end{bmatrix}$$

$$\boldsymbol{f}_2(\boldsymbol{x}) = \begin{bmatrix} -1 \\ 2x_1^2 + 3x_2^2 - 2 \end{bmatrix}$$



$$(\nabla s)^\top \boldsymbol{f}_1\Big|_{\boldsymbol{x}_0} \geq 0, \quad (\nabla s)^\top \boldsymbol{f}_2\Big|_{\boldsymbol{x}_0} \leq 0$$

Recap of conditions for existence and uniqueness of solutions of ODE

- Consider the ODE $\dot{x}(t) = f(x(t), t)$
- Questions:
  - Under which conditions does a solution exists?
  - Under which conditions the solution is unique?
- To answer both, the function $f()$ must be analyzed.
- But even more fundamentally: what does it mean that a function $x(t)$ is a solution of the the ODE?
- Accessible discussion in the online available book Trefethen, Lloyd N., Ásgeir Birkisson, and Tobin A. Driscoll. Exploring ODEs. Philadelphia: SIAM, 2017. http://people.maths.ox.ac.uk/trefethen/ExplODE/, chapters 3 and 11.

Classical solution (Peano, also Cauchy-Peano)

- $f(x(t), t)$ is continuous with respect to both x and t .

Strenghtening the requirement of continuity (Pickard-Lindelöf)

- $f(x(t),t)$ is continuous with respect to t but a stricter condition is imposed with respect to x – Lipschitz continuity.
- Not only existence but also uniqueness of a solution is guaranteed.
  - But similarly as with Peano conditions, here too the condition is not necessary, it is just sufficient – even if the function f is not Lipschitz continuous, there may exist a unique solution.
  - Since the condition is stricter than mere continuity, whatever goodies hold here too. In particular, the solution is guaranteed to be continuously differentiable.
  - If the function is only *locally Lipschitz*, the solution is guaranteed on some finite interval. If the function is (globally) Lipschitz, the solution is guaranteed on an unbounded interval.

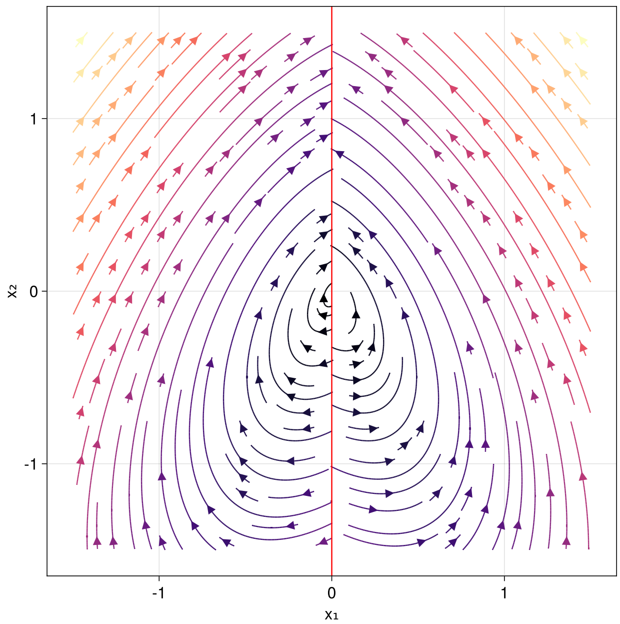
Extending the set of solutions (Carathéodory)

- In contrast with the classical solution, here the $x(t)$ can fail

Another example: nonexistence and nonuniqueness of solutions

The system with discontinuous RHS

$$\dot{x}_1 = -2x_1 - 2x_2 \,\mathrm{sgn}(\mathrm{x}_1),$$
$$\dot{x}_2 = x_2 + 4x_1 \,\mathrm{sgn}(\mathrm{x}_1)$$



can be reformulated as a switched system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; \boldsymbol{x} \leq 0$$
$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; \boldsymbol{x} > 0$$

Sliding mode dynamics (on simple boundaries)

- Attractive sliding mode at $\boldsymbol{x}_\mathrm{s}$, if there is a trajectory that ends at $\boldsymbol{x}_\mathrm{s}$, but no trajectory that starts at $\boldsymbol{x}_\mathrm{s}$.

Generalized solutions (Filippov)

- $x()$ is a Filippov solution on $[t_0, t_1]$ if for almost all t

$$\dot{x}(t) \in \overline{\text{co}}\{f(x(t), t)\},$$

- (Previous) example: $\mathcal{S}^+ = \{\boldsymbol{x} \mid x_1 = 0 \land x_2 \geq 0\}$

$$\begin{aligned}\dot{\boldsymbol{x}} &\in \overline{\text{co}}\{\boldsymbol{A}_1\boldsymbol{x}_1, \boldsymbol{A}_2\boldsymbol{x}_2\}\\ &= \alpha_1(t)\boldsymbol{A}_1\boldsymbol{x}_1 + \alpha_2(t)\boldsymbol{A}_2\boldsymbol{x}_2,\end{aligned}$$

  where $\alpha_1(t), \alpha_2(t) \geq 0, \alpha_1(t) + \alpha_2(t) = 1.$

- Not all weights keep the solution on $\mathcal{S}^+$.
  - We must have $\dot{x}_1 = 0$ for $\boldsymbol{x}(t) \in \mathcal{S}^+$:

$$\alpha_1(t)[-2x_1 + 2x_2] + \alpha_2(t)[-2x_1 - 2x_2] = 0$$

- Combining with $\alpha_1(t) + \alpha_2(t) = 1$:

$$\alpha_1(t) = \alpha_2(t) = 1/2.$$

- The dynamics on the sliding mode is

$$\dot{x}_1 = 0, \quad \dot{x}_2 = x_2, \quad \boldsymbol{x} \in \mathcal{S}^+.$$

Possible nonuniqueness on intersection of boundaries

Piecewise affine (PWA) systems

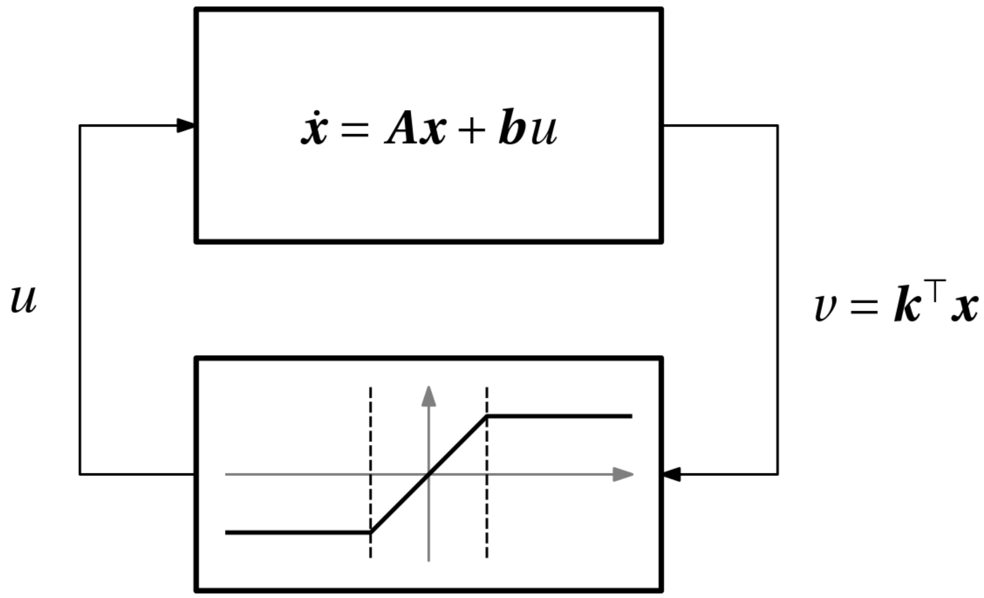
- Also piecewise linear (PWL) systems.
- Let's focus on state-driven switching only.
- Autonomous (no inputs)

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{b}_1, & \text{if } \boldsymbol{H}_1\boldsymbol{x} + \boldsymbol{g}_1 \leq 0, \\ \vdots \\ \boldsymbol{A}_m\boldsymbol{x} + \boldsymbol{b}_m, & \text{if } \boldsymbol{H}_m\boldsymbol{x} + \boldsymbol{g}_m \leq 0. \end{cases}$$

- Nonautonomous (with inputs)

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{B}_1 u + \boldsymbol{c}_1, & \text{if } \boldsymbol{H}_1\boldsymbol{x} + \boldsymbol{g}_1 \leq 0, \\ \vdots \\ \boldsymbol{A}_m\boldsymbol{x} + \boldsymbol{B}_m + \boldsymbol{c}_m, & \text{if } \boldsymbol{H}_m\boldsymbol{x} + \boldsymbol{g}_m \leq 0. \end{cases}$$

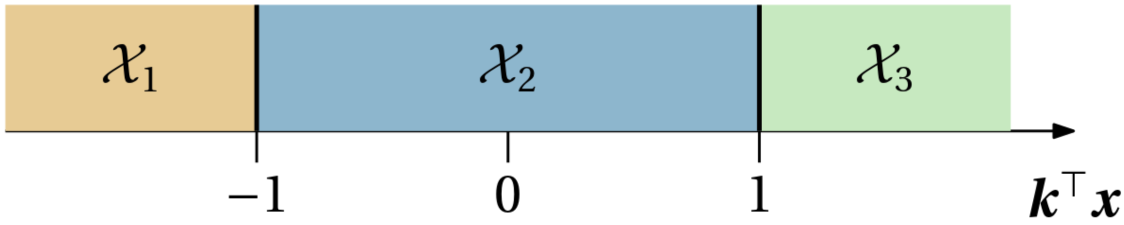
Example: Linear system with saturated state feedback



$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\,\mathrm{sat}(v), \quad v = \boldsymbol{k}^T \boldsymbol{x}.$$

Piecewise affine dynamics

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}, & \text{if } \boldsymbol{x} \in \mathcal{X}_1, \\ (\boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^\top)\boldsymbol{x}, & \text{if } \boldsymbol{x} \in \mathcal{X}_2, \\ \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}, & \text{if } \boldsymbol{x} \in \mathcal{X}_3, \end{cases}$$



$$\mathcal{X}_1 = \{\boldsymbol{x} \mid \boldsymbol{H}_1\boldsymbol{x} + g_1 \leq 0\}$$
$$\mathcal{X}_2 = \{\boldsymbol{x} \mid \boldsymbol{H}_2\boldsymbol{x} + \boldsymbol{g}_2 \leq 0\}$$
$$\mathcal{X}_3 = \{\boldsymbol{x} \mid \boldsymbol{H}_3\boldsymbol{x} + g_3 \leq 0\}$$

$$\boldsymbol{H}_1 = \boldsymbol{k}^\top, \quad g_1 = 1,$$
$$\boldsymbol{H}_2 = \begin{bmatrix} -\boldsymbol{k}^\top \\ \boldsymbol{k}^\top \end{bmatrix}, \quad \boldsymbol{g}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$
$$\boldsymbol{H}_3 = -\boldsymbol{k}^\top, \quad g_3 = 1.$$

Approximation of smooth systems

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2|x_2| - x_1(1 + x_1^2) \end{bmatrix}$$

Software for PWA modelling and analysis

- PLECS
  - Power electronics
  - Commercial
- Multiparametric Toolbox (MPT) for Matlab
  - General
  - FOSS