Output LQR, LQG and $\mathcal{H}_2$ optimal control

Graduate course on Optimal and Robust Control (spring'20)

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Lecture outline

Guaranteed stability margins for LQ optimal regulator

Stochastic LQ optimal regulator

Output LQR

Optimal LQG output feedback—state feedback and estimator
- Optimal estimation of states
- LQG optimal control

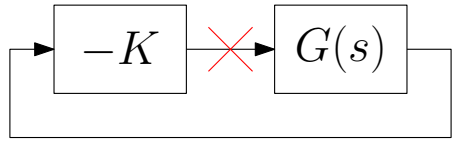
LQG as $\mathcal{H}_2$-optimal control



Guaranteed stability margins for LQ optimal regulator

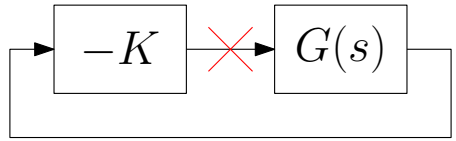


Guaranteed stability margins for LQ optimal regulator





Guaranteed stability margins for LQ optimal regulator



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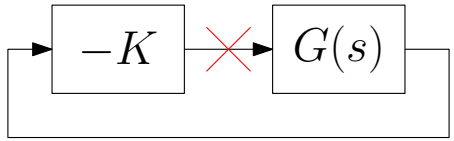
Guaranteed stability margins for LQ optimal regulator



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Open-loop transfer function

$$\mathbf{L}(s) = \mathbf{K}\mathbf{G}(s) = \mathbf{K}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$



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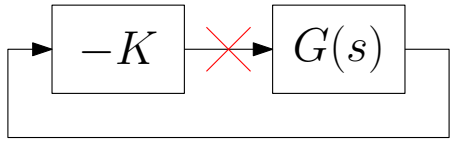

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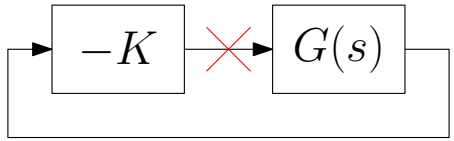

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Return difference $1 + L(s)$ on the imaginary axis



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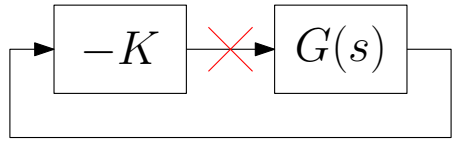

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Magnitude $|1 + L(j\omega)| = |-1 - L(j\omega)| \quad \forall\omega \in \mathbb{R}$.



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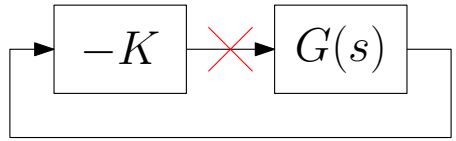

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Easier to evaluate

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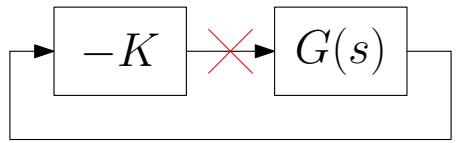
Guaranteed stability margins for LQ optimal regulator



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$$|1 + L(j\omega)|^2 = \overline{(1 + L(j\omega))}(1 + L(j\omega))$$



Conjugate system



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For MIMO systems

$$\mathbf{L}^*(s) = \mathbf{L}^{\mathsf{T}}(-s)$$



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Exactly what Matlab does with **L'**



Kalman's identity for return difference for LQR



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For $\mathbf{R} = \rho I$



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Kalman's identity for return difference for LQR

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$$(\mathbf{I} + \mathbf{L}^*)(\mathbf{I} + \mathbf{L}) = \mathbf{I} + \frac{1}{\rho}\mathbf{B}^{\mathsf{T}}(-sI - \mathbf{A})^{-\mathsf{T}}\mathbf{Q}(sI - \mathbf{A})^{-1}\mathbf{B}$$



Kalman's identity for return difference for LQR

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For SISO systems

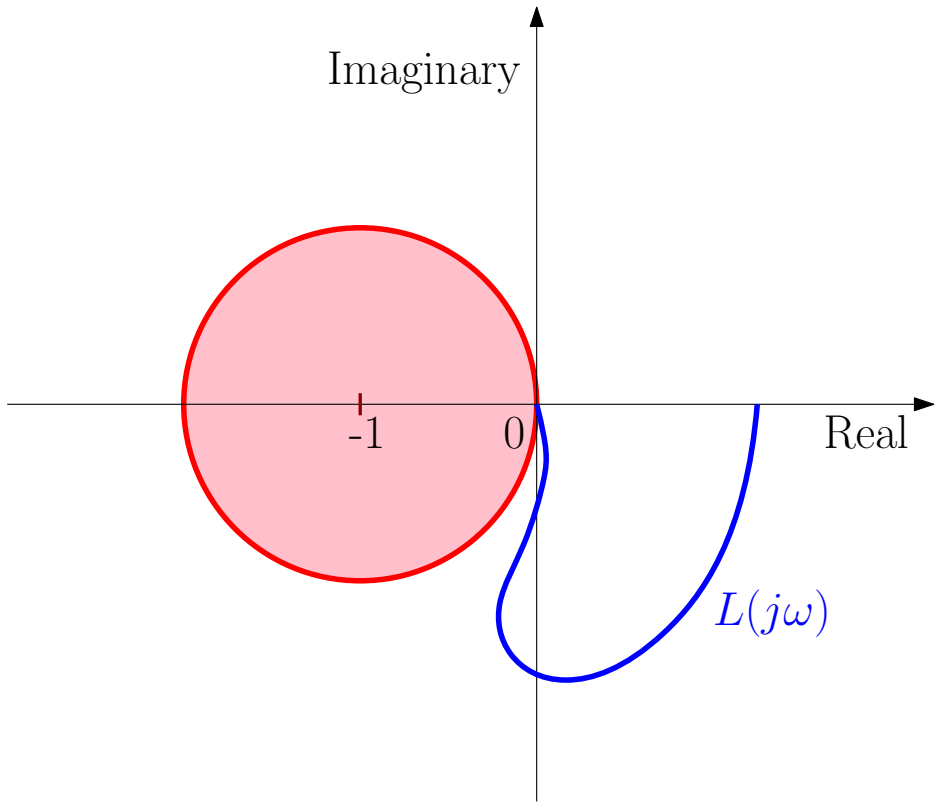
$$\boxed{|1 + KG(j\omega)| \geq 1, \quad \forall \omega}$$



Stability margins



Stability margins





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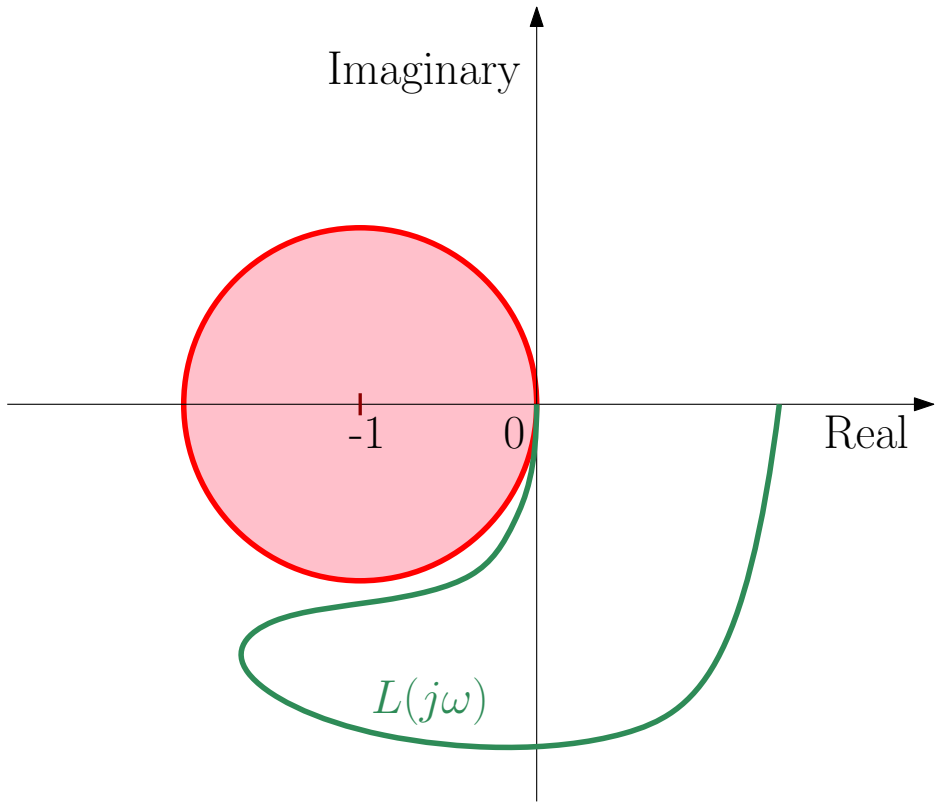



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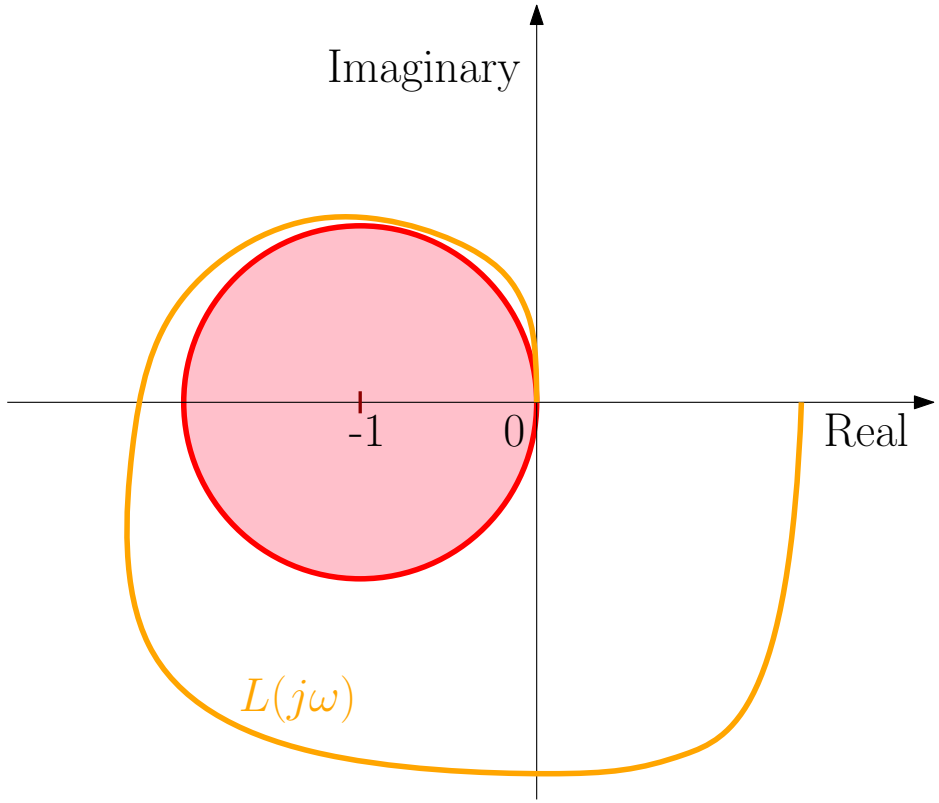



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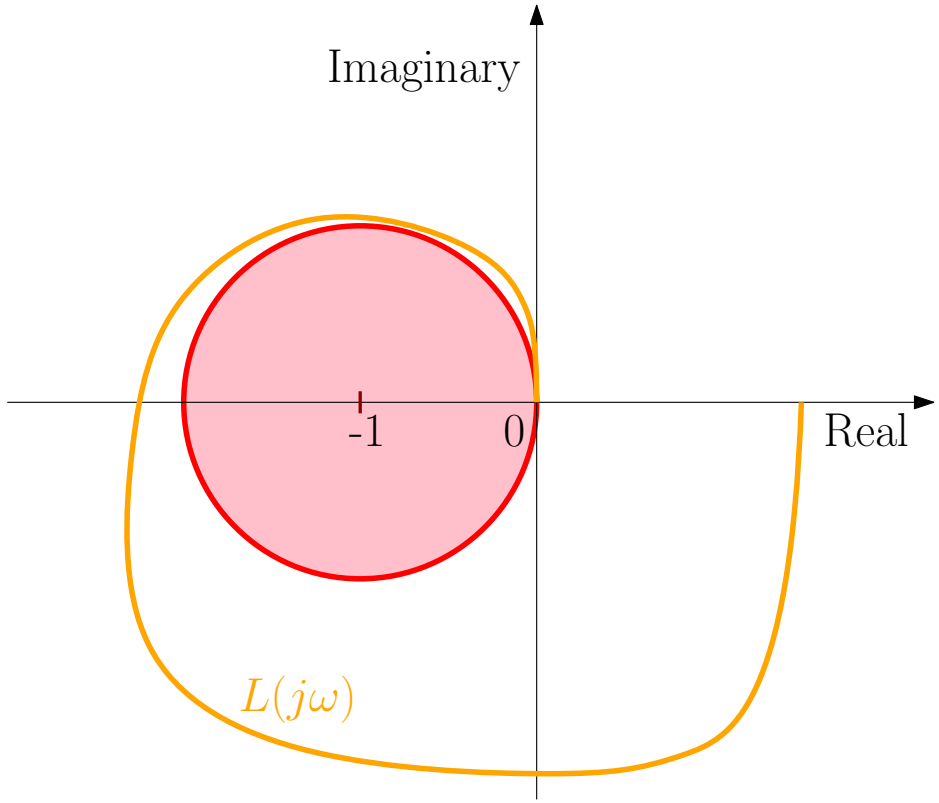

$$GM_+ = \infty, \quad GM_- = \frac{1}{2}, \quad PM_- \leq -60^\circ, \quad PM_+ \geq 60^\circ$$



Run a few times in Matlab

```
n = 5;                      % order of the plant
G = rss(n,n,1);             % random system, 1 input, n outputs
G.c = eye(n,n);             % all state variables as outputs
G.d = zeros(n,1);           % zero feedthrough
Q = eye(n,n);               % penalty on x
R = 1;                      % penalty on u
[K,S,e] = lqr(G,Q,R)        % LQR design
L = K*G;                    % open-loop transfer function
margin(L)
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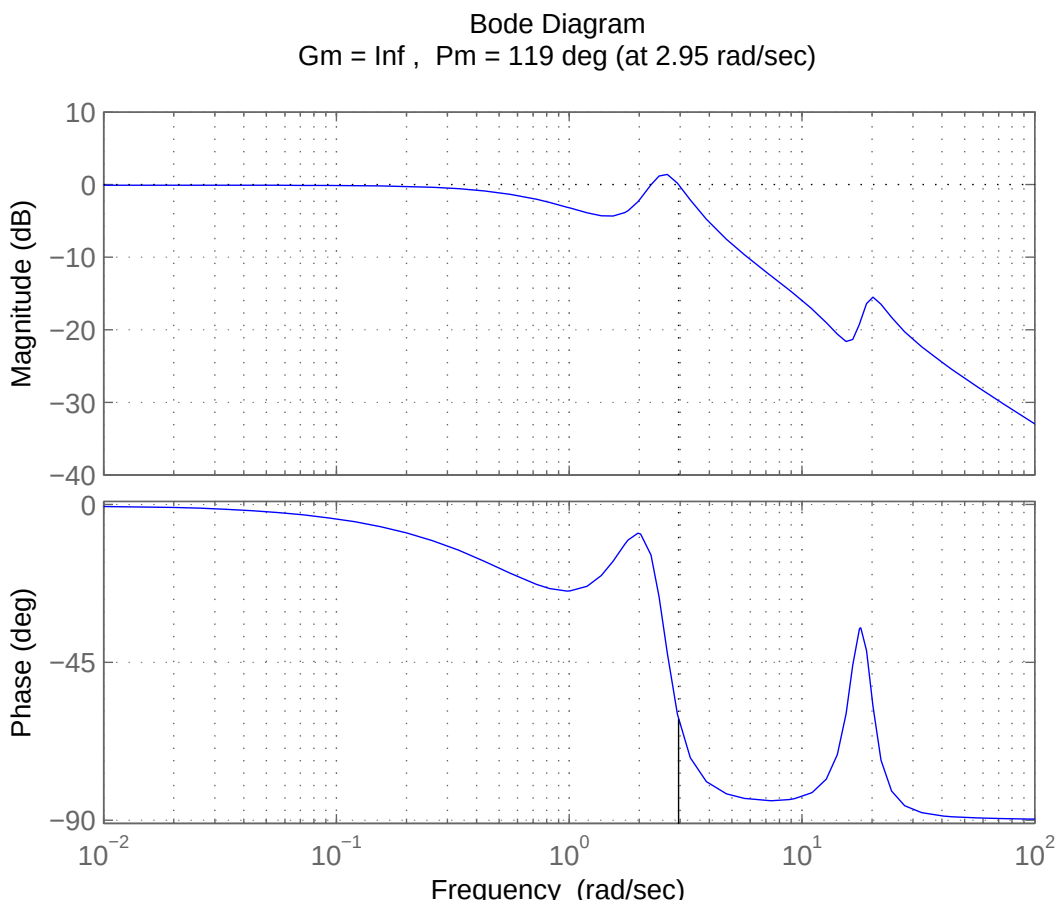

Stochastic LQ optimal regulator



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Plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$



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$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_w\mathbf{w}(t), \qquad \mathbf{x}(0) = \dots$$



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Random initial states: zero mean and a covariance matrix

$$\mathbf{E}\left\{\mathbf{x}(0)\right\} = \mathbf{0},$$
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Disturbace: white noise with spectral density $\mathbf{S}_w$ and uncorellated with the initial state.

$$\mathbf{E}\left\{\mathbf{w}(t)\right\} = \mathbf{0},$$
$$\mathbf{E}\left\{\mathbf{w}(t)\mathbf{w}^{\mathsf{T}}(t+\tau)\right\} = \mathbf{S}_w\delta(\tau)$$



The cost function

$$J = \frac{1}{2}\mathbf{E}\left\{\mathbf{x}^{\mathsf{T}}(t_{\mathbf{f}})\mathbf{S}\mathbf{x}(t_{\mathbf{f}}) + \int_0^{t_{\mathbf{f}}} \left[\mathbf{x}^{\mathsf{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathsf{T}}(t)\mathbf{R}\mathbf{u}(t)\right] dt\right\}$$



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Scaling by $\frac{2}{t_{\mathbf{f}}}$

$$\begin{aligned} J' &= \frac{1}{2}\mathbf{E}\left\{\lim_{t_{\mathbf{f}}\to\infty} \frac{2}{t_{\mathbf{f}}} \int_0^{t_{\mathbf{f}}} \left[\mathbf{x}^{\mathsf{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathsf{T}}(t)\mathbf{R}\mathbf{u}(t)\right] dt\right\} \\ &= \mathbf{E}\left\{\mathbf{x}^{\mathsf{T}}(\infty)\mathbf{Q}\mathbf{x}(\infty) + \mathbf{u}^{\mathsf{T}}(\infty)\mathbf{R}\mathbf{u}(\infty)\right\} \end{aligned}$$



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The steady-state covariance matrix for the state vector can be found by solving the Lyapunov equation

$$\boxed{(\mathbf{A}-\mathbf{BK})\mathbf{\Sigma}_x(\infty)+\mathbf{\Sigma}_x(\infty)(\mathbf{A}-\mathbf{BK})^{\mathsf{T}}+\mathbf{B}_w\mathbf{S}_w\mathbf{B}_w^{\mathsf{T}}=0}$$



Ex.: Stochastic LQR for a satellite tracking antenna

Pointing antenna subject to random wind torque

$$\begin{bmatrix} \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \underbrace{\begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}}_{\mathbf{x}} + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} w(t)$$

where $\theta(t)$ is a pointing error $[deg]$, $u(t)$ is a control torque $[Nm]$ and $w(t)$ is a random wind toque $[Nm]$.



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The wind torque is modelled as white noise with a spectral density $S_w = 5000 N^2 m^2 / Hz$.
Cost function

$$J = \mathbf{E}\left[\begin{bmatrix} \theta(\infty) & \dot{\theta}(\infty) \end{bmatrix} \begin{bmatrix} 180 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(\infty) \\ \dot{\theta}(\infty) \end{bmatrix} + u^2(\infty) \right]$$



LQR in Matlab

```
A = [0  1; 0  -0.1];
Bu = [0; 0.001];
Bw = [0; 0.001];

Sw = 5000;

Q = [180  0; 0  0];
R = 1;

[S,D,K] = care(A,Bu,Q,R);

Sx = lyap(A-Bu*K,Bw*Sw*Bw')
```



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```
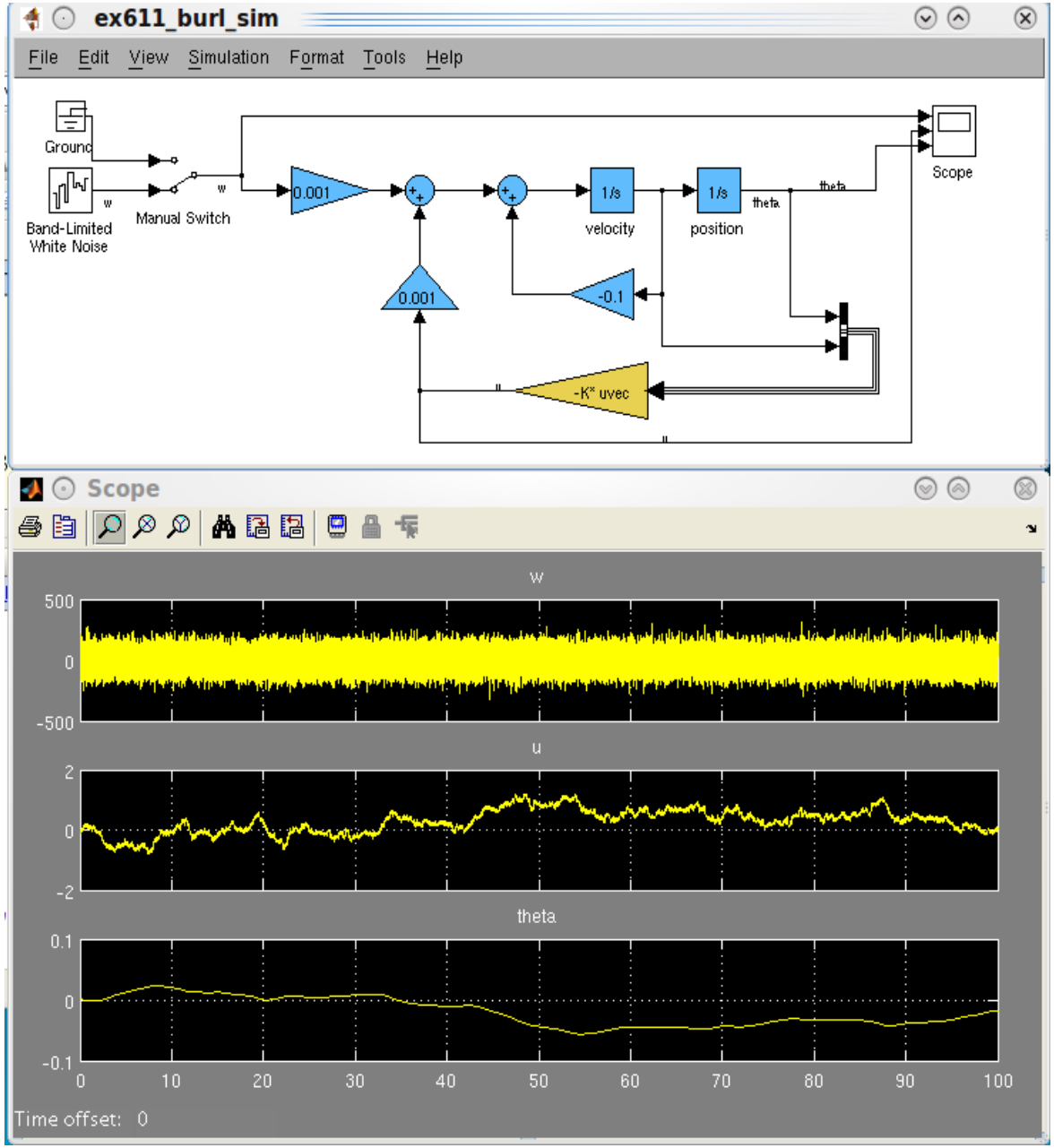

$$\mathbf{\Sigma}_x(\infty) = \begin{bmatrix} 0.9854\text{deg}^2 & 0 \\ 0 & 0.1141\frac{\text{deg}^2}{s^2} \end{bmatrix}$$




ex611_burl_sim
File Edit View Simulation Format Tools Help
Ground
Band-Limited White Noise
w
Manual Switch
w
0.001
1/s
velocity
1/s
position
theta
theta
Scope
0.001
-0.1
-K* uvec
u
Scope
w
500
0
-500
u
2
0
-2
theta
0.1
0
-0.1
0 10 20 30 40 50 60 70 80 90 100
Time offset: 0

Alternative input-output interpretation of stochastic LQR



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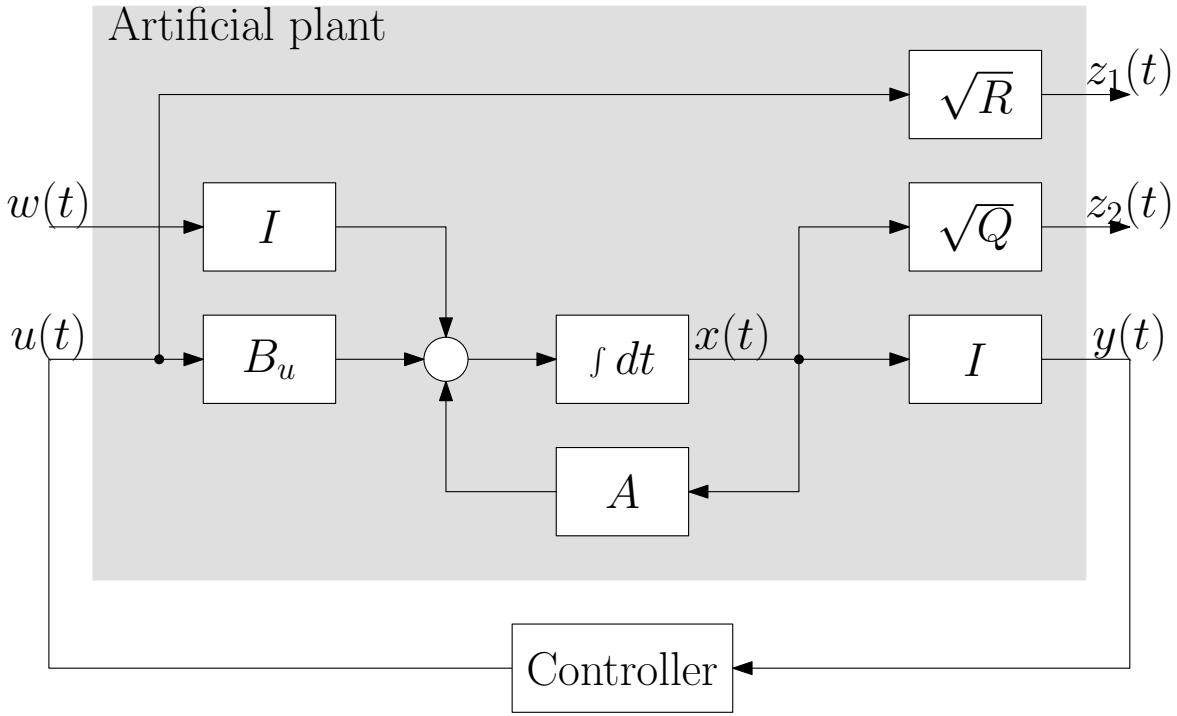
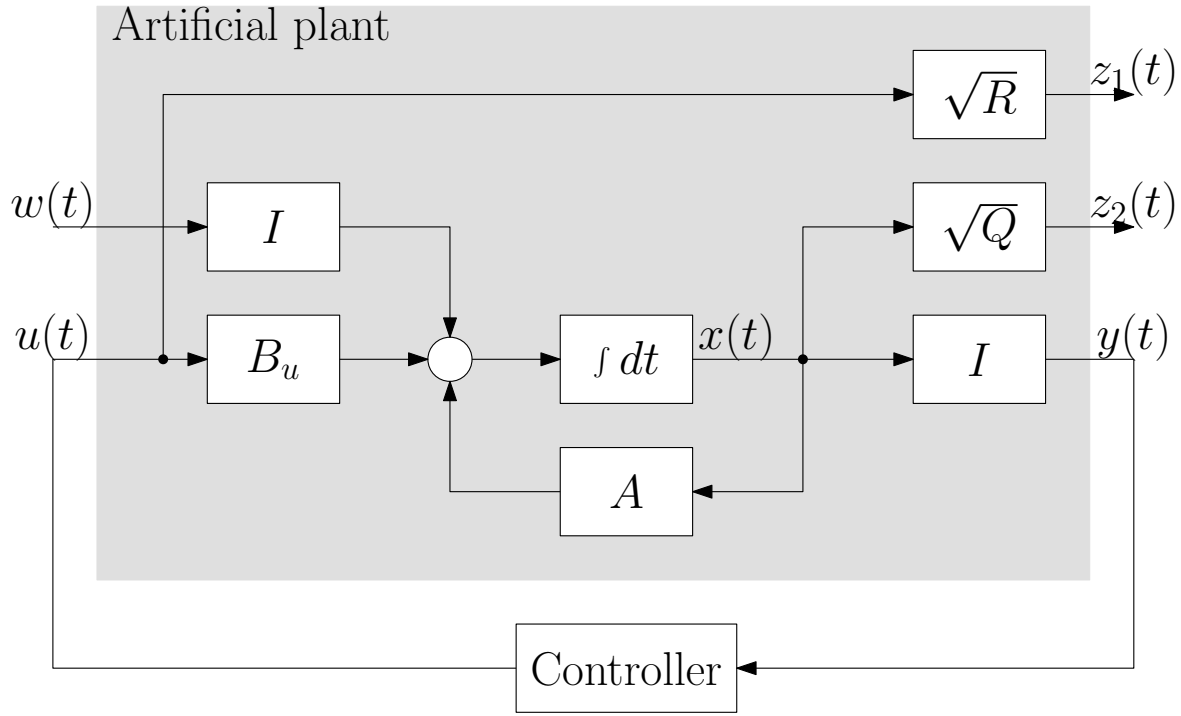


Alternative input-output interpretation of stochastic LQR



$$J = \mathbf{E}\left\{ \begin{bmatrix} \mathbf{z}_1^{\mathsf{T}}(\infty) & \mathbf{z}_2^{\mathsf{T}}(\infty) \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(\infty) \\ \mathbf{z}_2(\infty) \end{bmatrix} \right\}$$



$\mathcal{H}_2$ system norm



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For a stable and strictly proper LTI system G

$$\|\mathbf{G}\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left[\mathbf{G}^*(j\omega)\mathbf{G}(j\omega)\right] d\omega}$$



$\mathcal{H}_2$ system norm

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By using Parseval's theorem (relating inner products in time and frequency domains)

$$\boxed{\|\mathbf{G}\|_2 = \sqrt{\int_{0}^{\infty} \operatorname{tr}\left[\mathbf{g}^{\mathsf{T}}(t)\mathbf{g}(t)\right] dt}}$$



$\mathcal{H}_2$ norm as a gain of a system subject to a stationary white noise input



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$$\boxed{\mathbf{E}\left\{\mathbf{y}^{\mathsf{T}}(\infty)\mathbf{y}(\infty)\right\} = \|\mathbf{G}\|_2^2 \, S_w}$$



$\mathcal{H}_2$ norm as a gain of a system subject to a stationary white noise input

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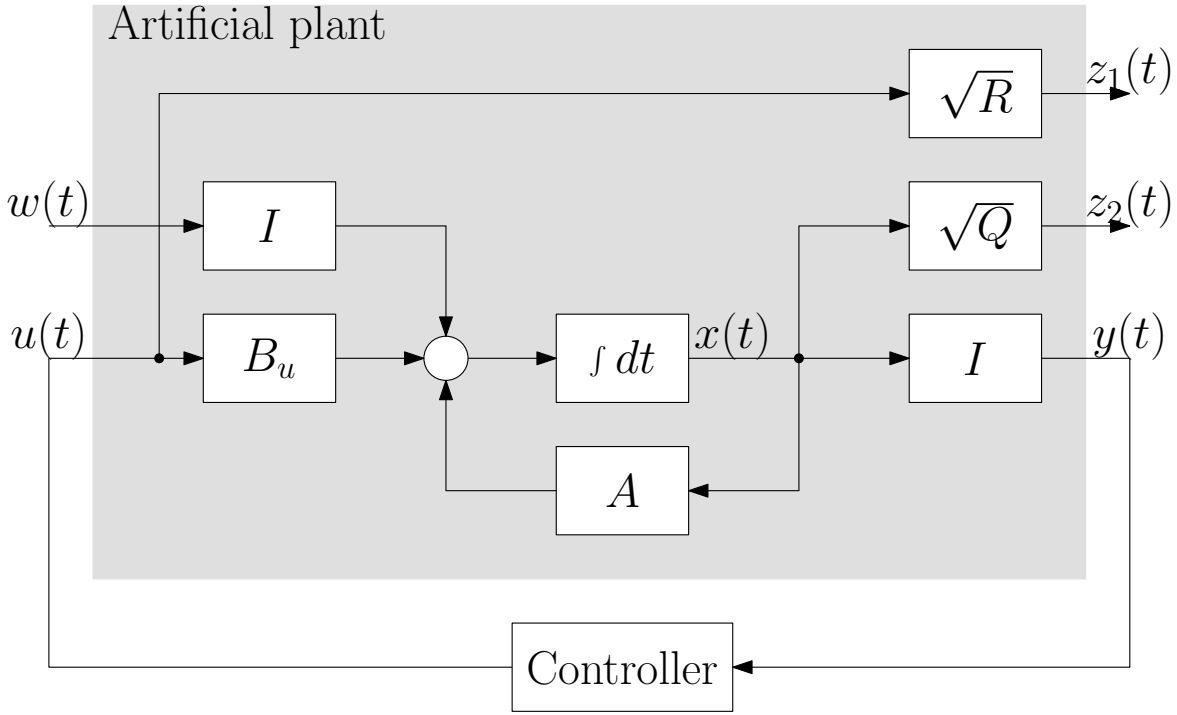
See, for example, Doyle, Francis, Tannenbaum (1990) at `https://www.control.utoronto.ca/people/profs/francis/dft.html`.



$\mathcal{H}_2$-optimal control



$\mathcal{H}_2$-optimal control



Find a stabilizing controller that minimizes $\mathcal{H}_2$ norm of the closed loop system.



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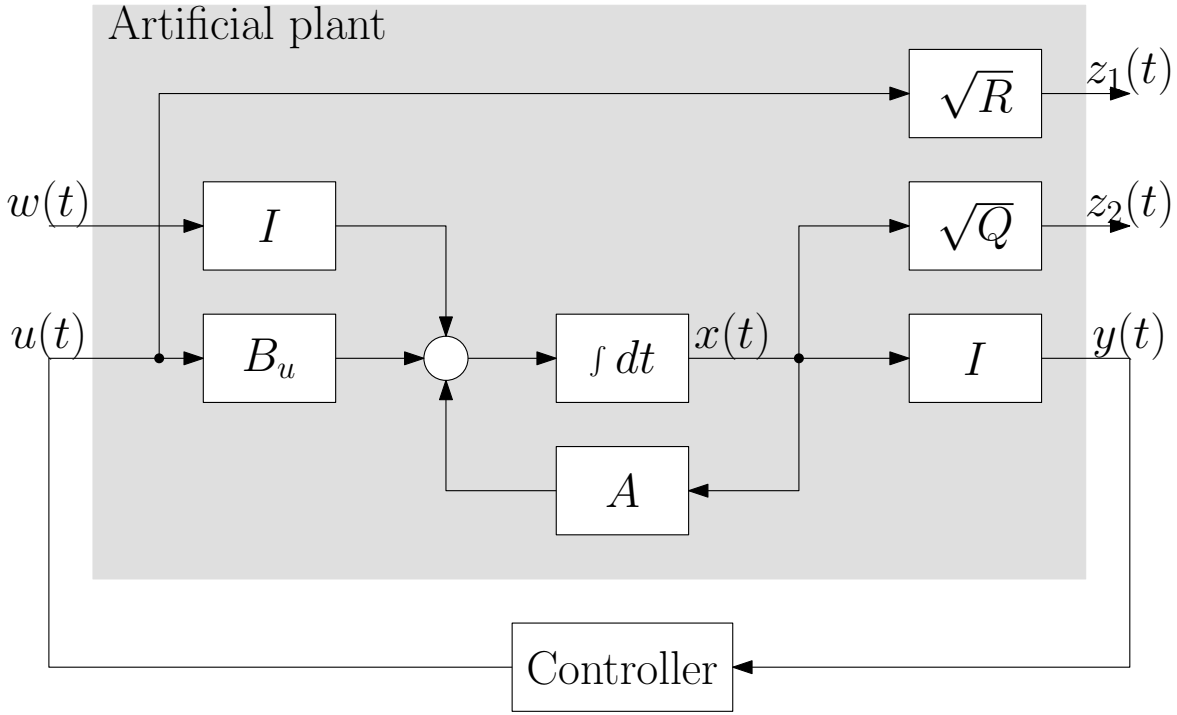

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Matlab: **h2syn()** from Robust Control Toolbox



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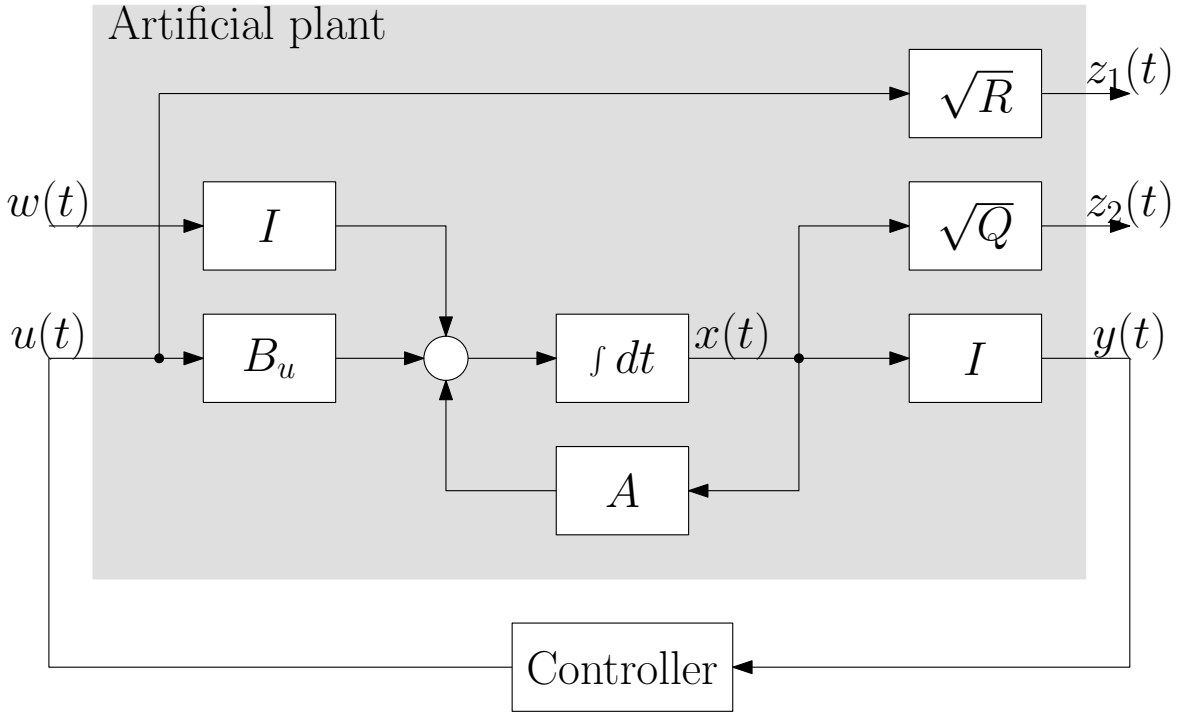

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Can easily extend to output feedback control, nonwhite noise, ...



Output feedback LQR



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Not all the states are measured:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$
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Generally nonconvex numerical optimization.



Optimal LQG output feedback control - state feedback and estimator

Combination of LQ-optimal state feedback and optimal estimator (Kalman filter).



Estimator (observer) of the states



Estimator (observer) of the states

Plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}$$



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$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \underbrace{\mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}})}_{\text{correction}}$$



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$$\boxed{\dot{\hat{\mathbf{x}}} = \underbrace{(\mathbf{A} - \mathbf{L}\mathbf{C})}_{\mathbf{A}_o}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{y}}$$



Kalman filter



Kalman filter

Plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_w\mathbf{w}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t)$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are white noises with spectral densities $\mathbf{S}_w$ and $\mathbf{S}_v$, respectively.



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Matlab: **kalman()**



Example: Kalman filter



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Estimate the range and radial velocity of an aircraft from noisy radar measurements.



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Model

$$\begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

where $r(t)$ is the actual range of the aircraft.



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Initial conditions

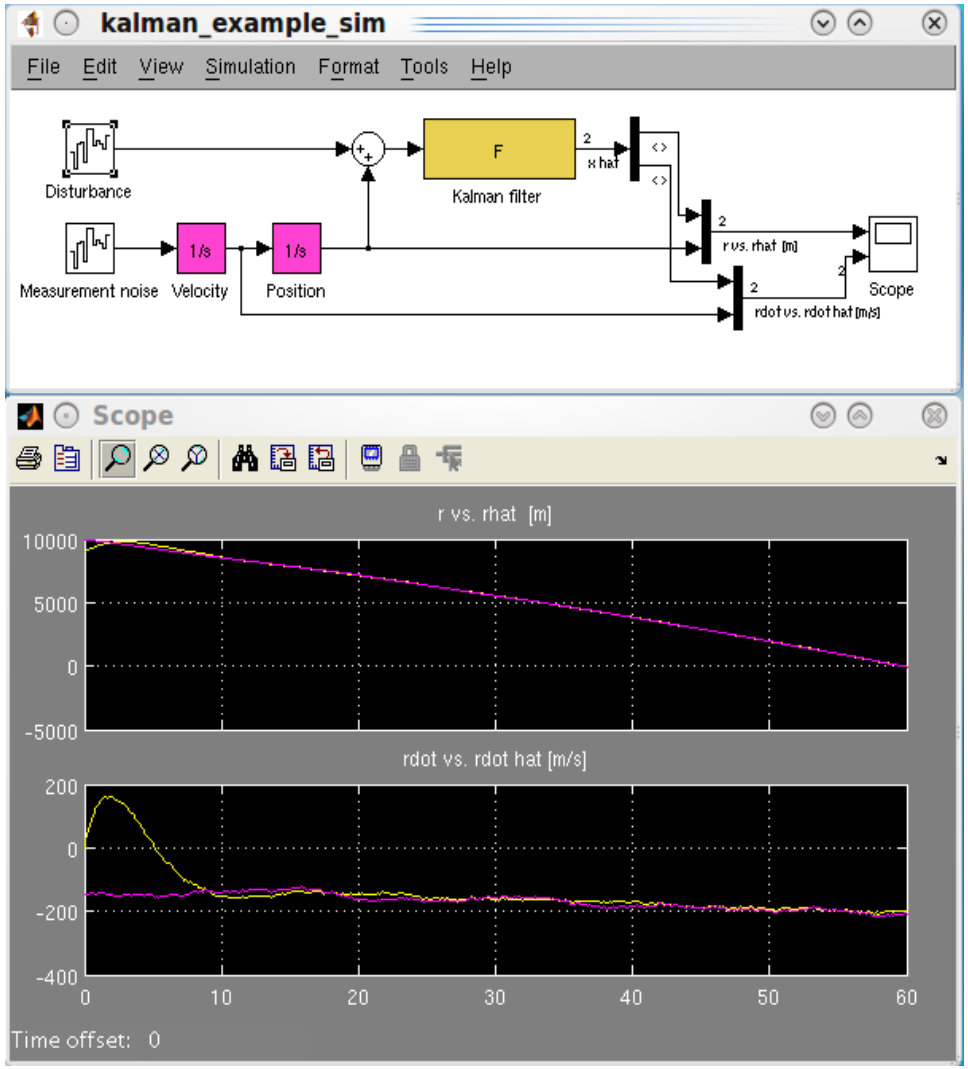
$$\begin{bmatrix} r(0) \\ \dot{r}(0) \end{bmatrix} = \begin{bmatrix} 10000m \\ -150m/s \end{bmatrix}$$



$$S_w = 4\frac{m^2}{s^4 Hz}$$

$$S_v = 10^4\frac{m^2}{Hz}$$

$$\Sigma_e(0) = \begin{bmatrix} 10^6 m^2 & 0 \\ 0 & 4\times 10^5 m^2/s^2 \end{bmatrix}$$



Combined Kalman filtering and LQ optimal control: LQG optimal control



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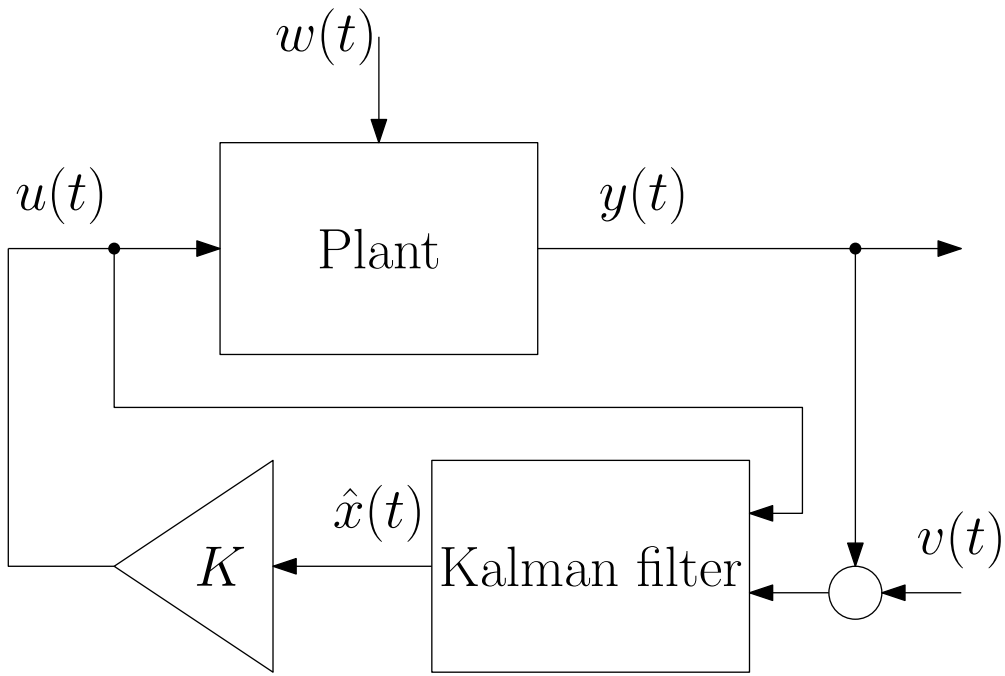



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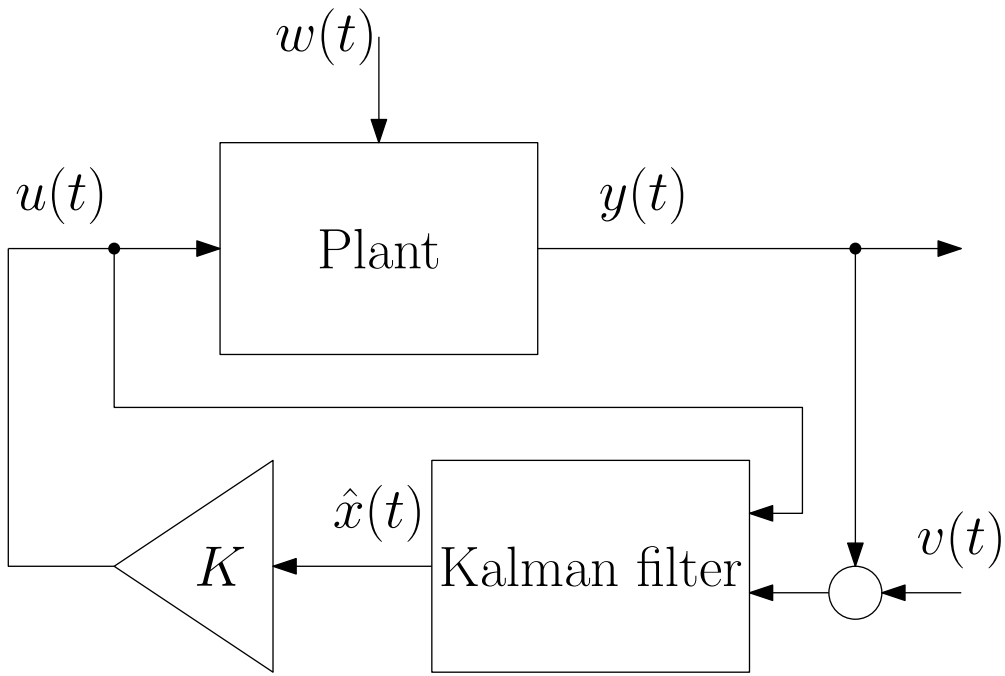

Numerical solution: 2 AREs



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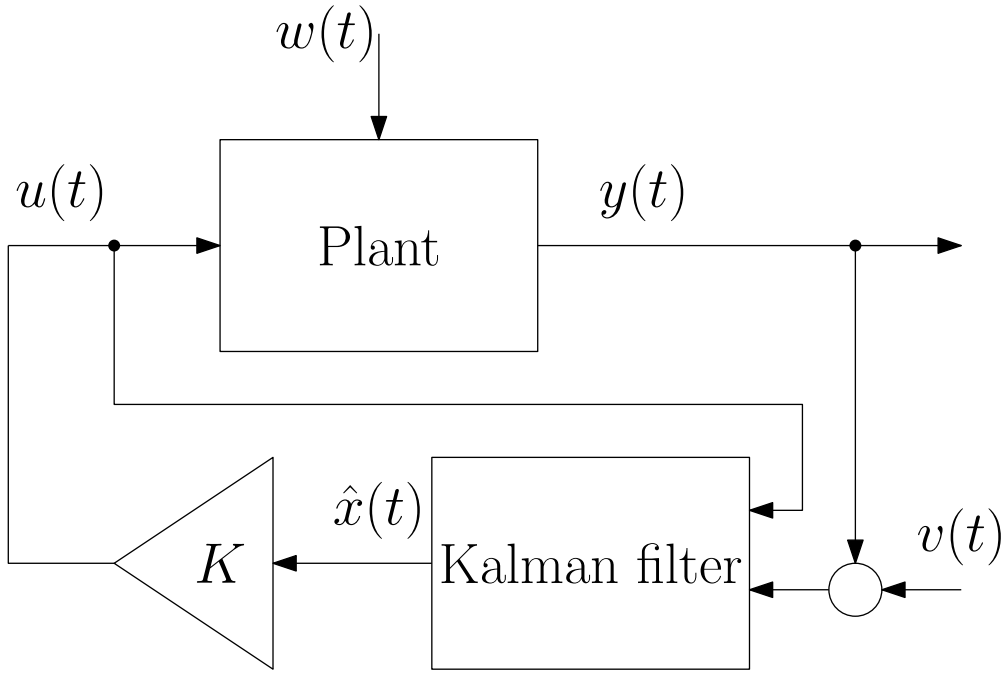


Numerical solution: 2 AREs

Matlab: **lqgreg()** combines the results of **lqr()** (or **dlqr()**) and **kalman()**



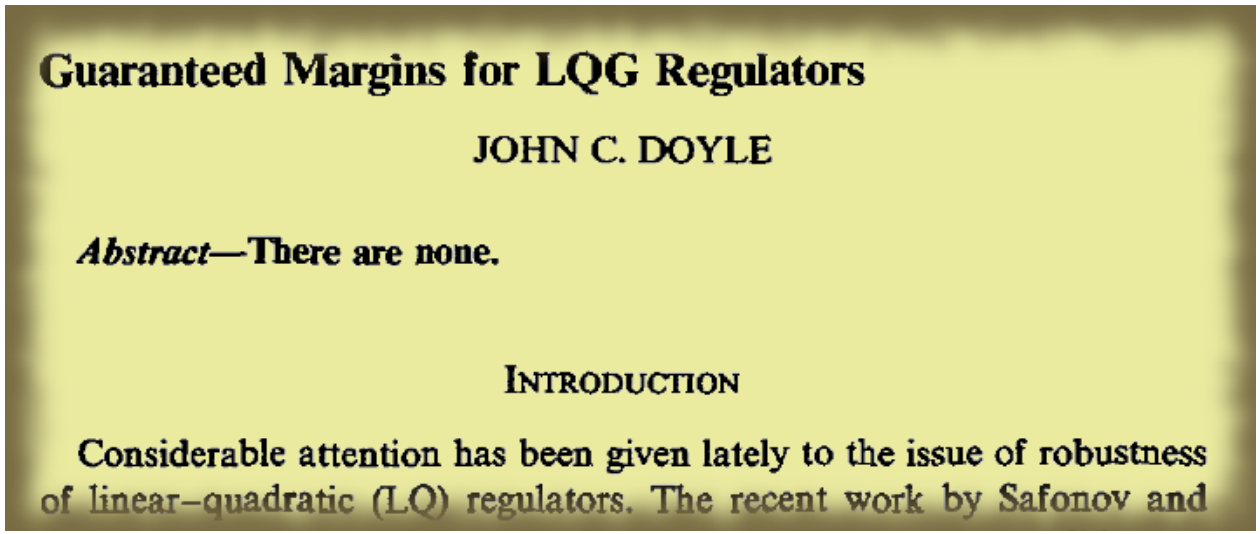
Example: satellite tracking antenna with noisy measurements of angle



Stability margins of LQG



Stability margins of LQG



Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

***Abstract*—There are none.**

INTRODUCTION

Considerable attention has been given lately to the issue of robustness of linear–quadratic (LQ) regulators. The recent work by Safonov and

J. Doyle, "Guaranteed margins for LQG regulators," IEEE Transactions on Automatic Control, vol. 23, no. 4, pp. 756–757, Aug. 1978, doi: 10.1109/TAC.1978.1101812.



Example [Doyle 1978]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t),$$

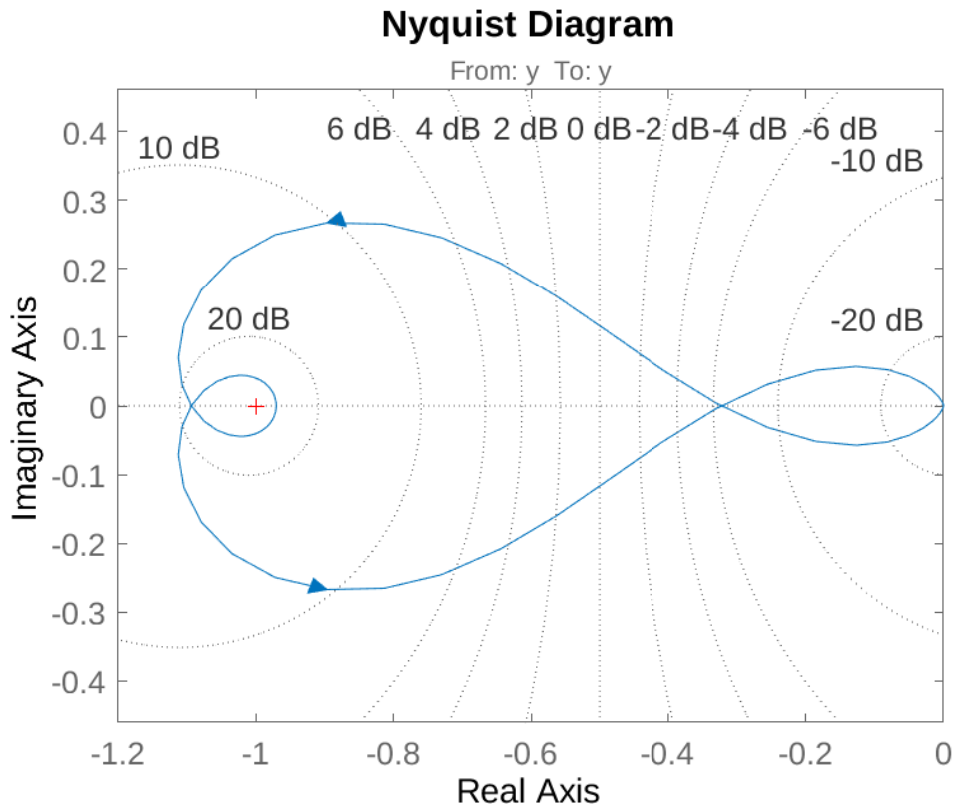
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) + v(t)$$

$$\mathbf{Q} = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix},$$

$$R = 1,$$

$$S_w = 10,$$

$$S_v = 1$$





LTR control - Heuristic way to improve robustness of LQG



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Make it similar to LQR $\Longrightarrow$ make the Kalman filter rely less on u



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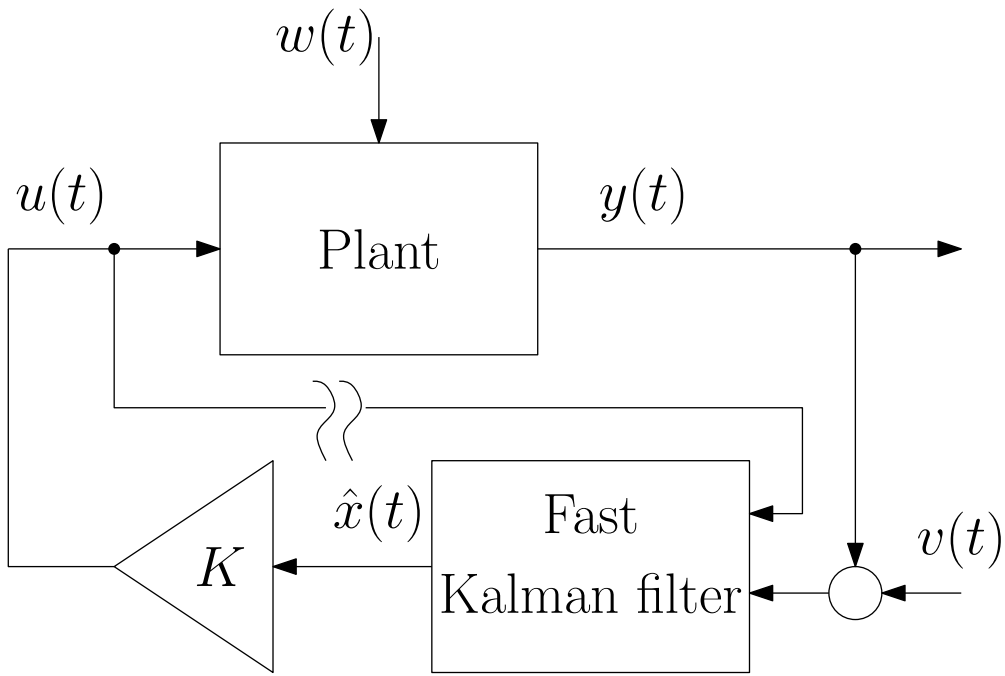

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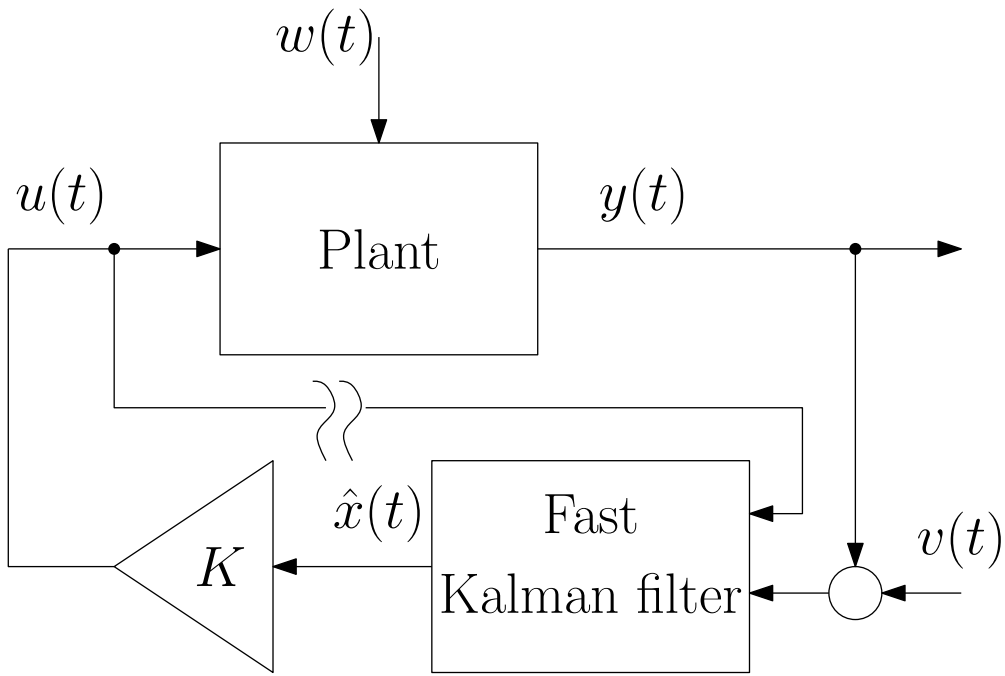

But how?

Introduce fictitious noise $w_f(t)$ during the design (not in reality) that enters the system in the same way as the control signal



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$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_w\mathbf{w}(t) + \mathbf{B}_u\mathbf{w}_f(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \begin{bmatrix}\mathbf{B}_w & \mathbf{B}_u\end{bmatrix}\begin{bmatrix}\mathbf{w}(t) \\ \mathbf{w}_f(t)\end{bmatrix}\end{aligned}$$



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Spectral density of the white noise disturbance is then

$$\mathbf{S} = \begin{bmatrix}\mathbf{S}_w & 0 \\ 0 & S_{w_f} I\end{bmatrix}$$



Ex.: LQG/LTR for a robot arm



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Single link robotic arm is to be held vertical

$$J\ddot{\theta}(t) = mg \sin \theta(t) + u(t)$$



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where w is a random disturbance by a torquer circuit with a spectral density $S_w = 1$.



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where the measurement noise spectral density is $S_v = 1$.



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where the measurement noise spectral density is $S_v = 1$. Steady state LQG controller should minimizes

$$J = \mathbf{E}\left\{\theta^2 + 16u^2\right\}$$



LQG as $\mathcal{H}_2$-optimal control



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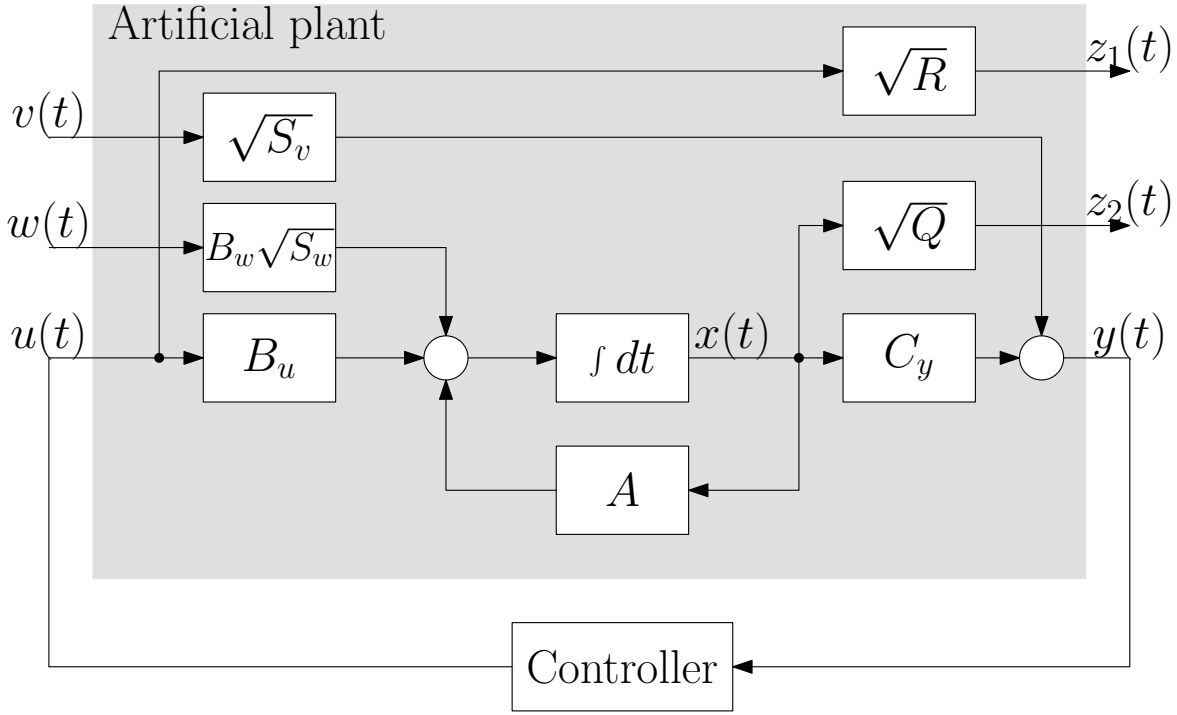
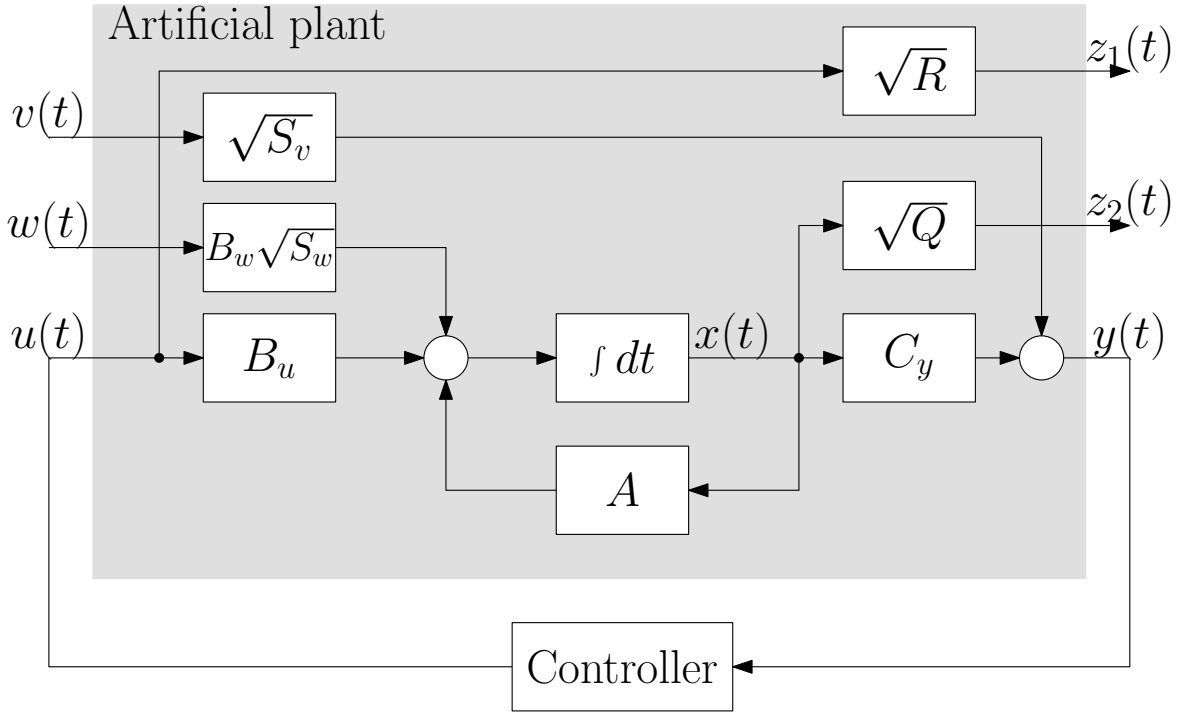
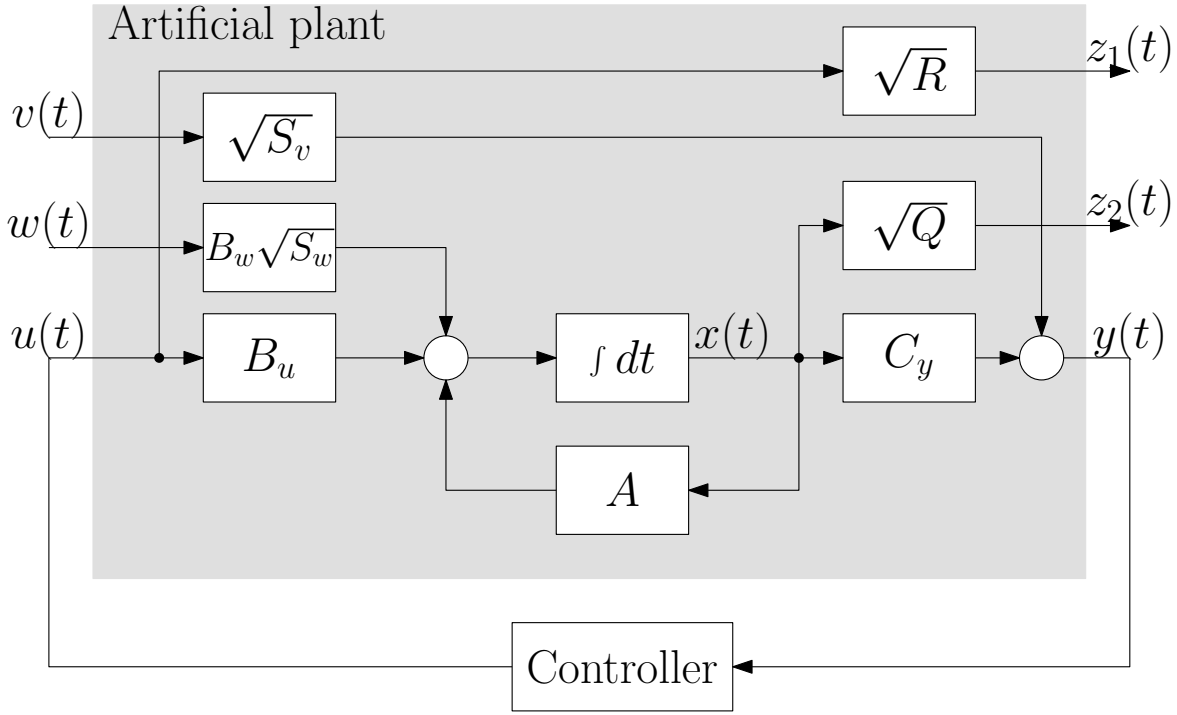


LQG as $\mathcal{H}_2$-optimal control



Generalized plant

LQG as $\mathcal{H}_2$-optimal control



Generalized plant

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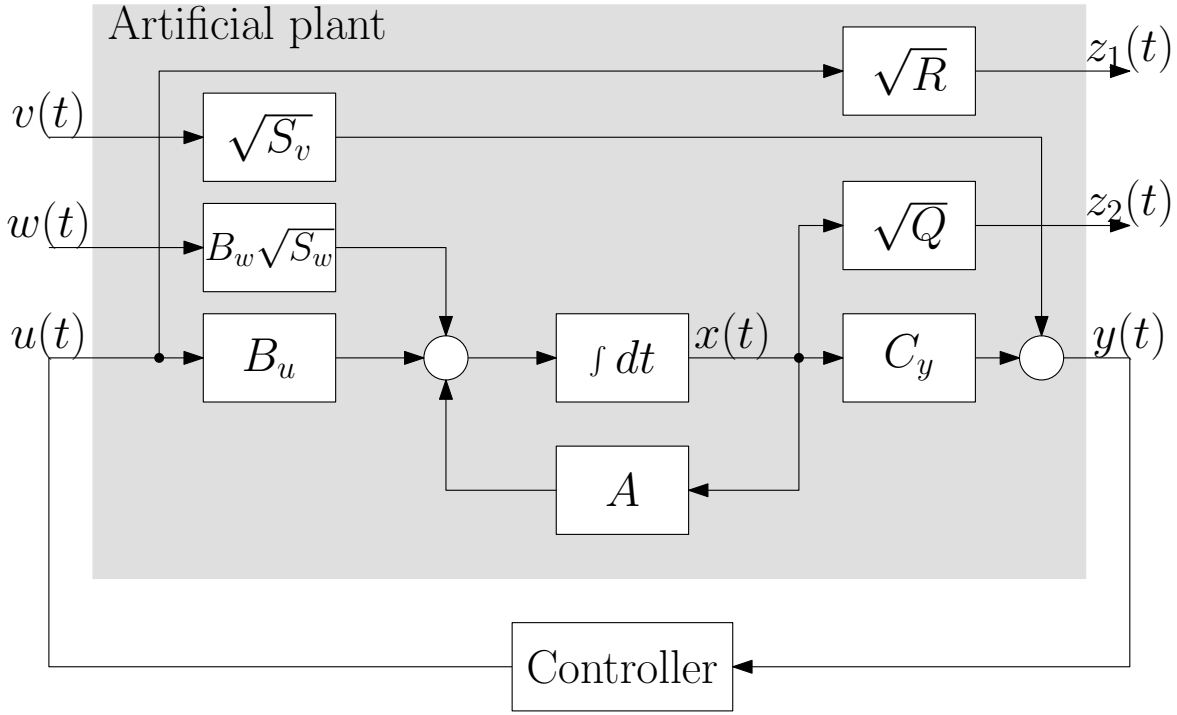
LQG as $\mathcal{H}_2$-optimal control



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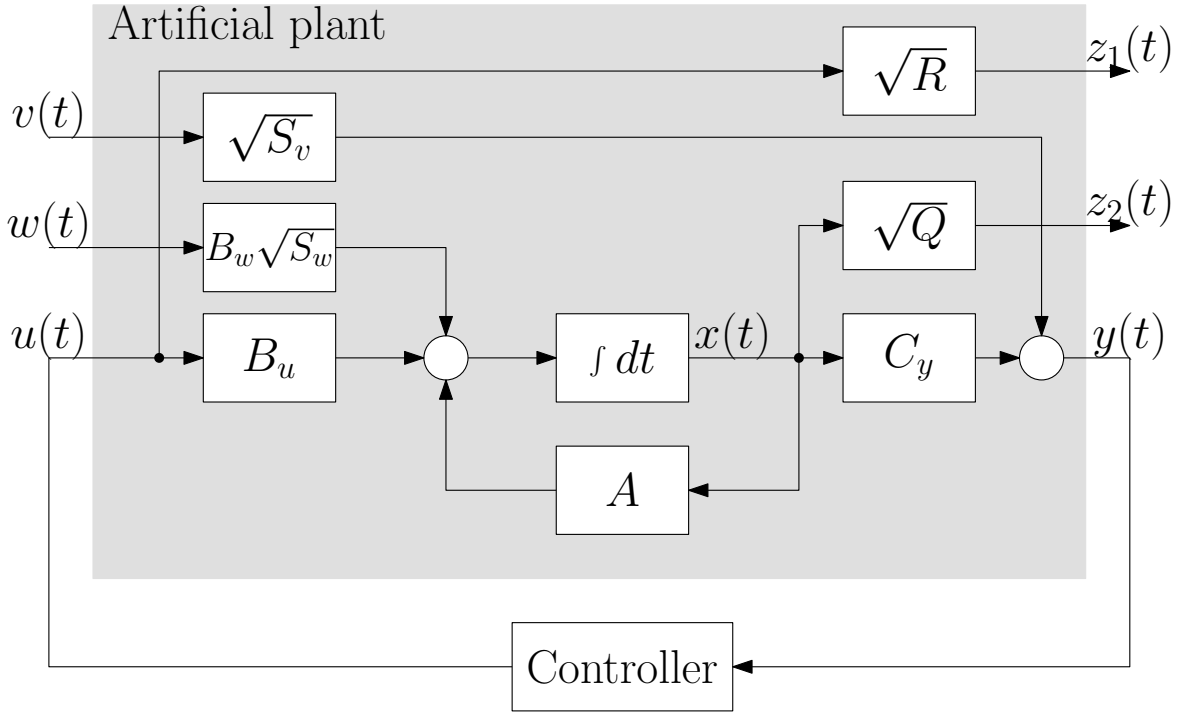

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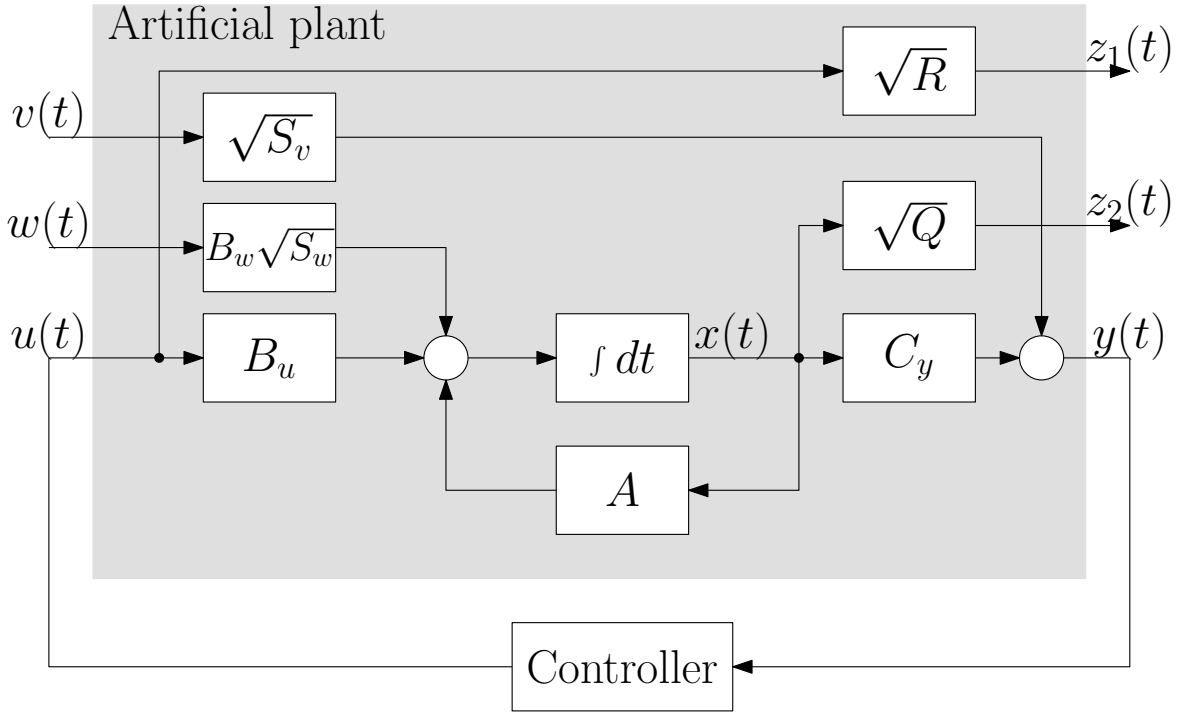

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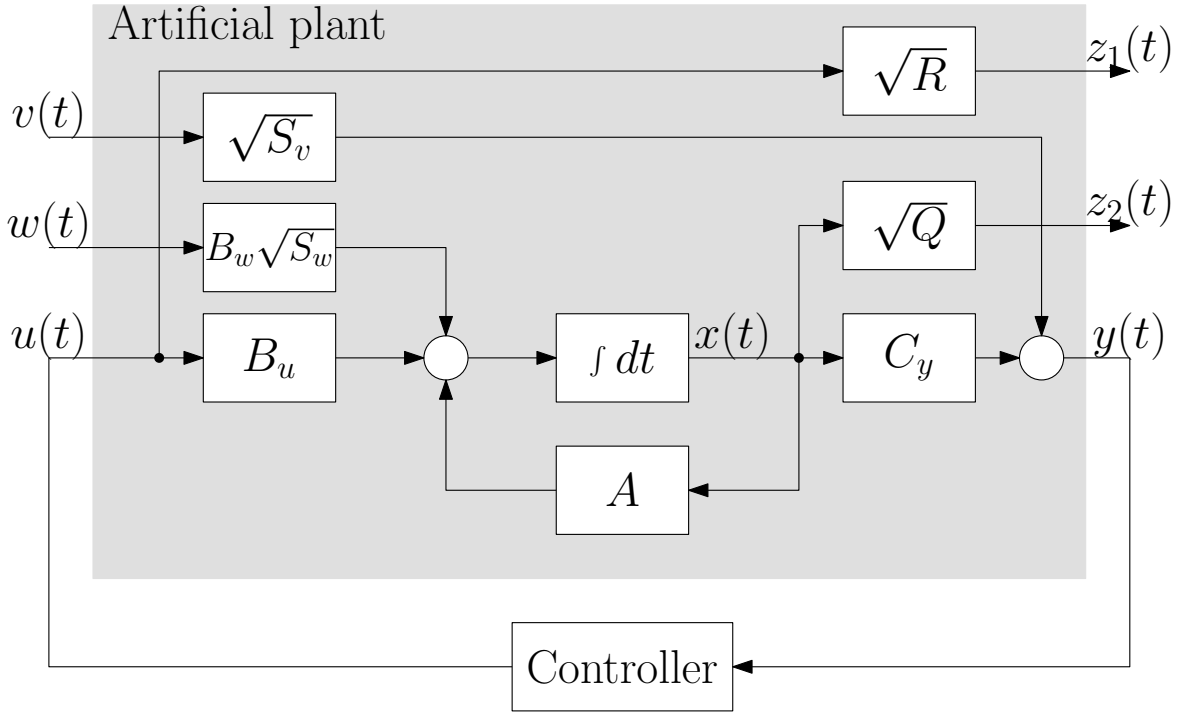

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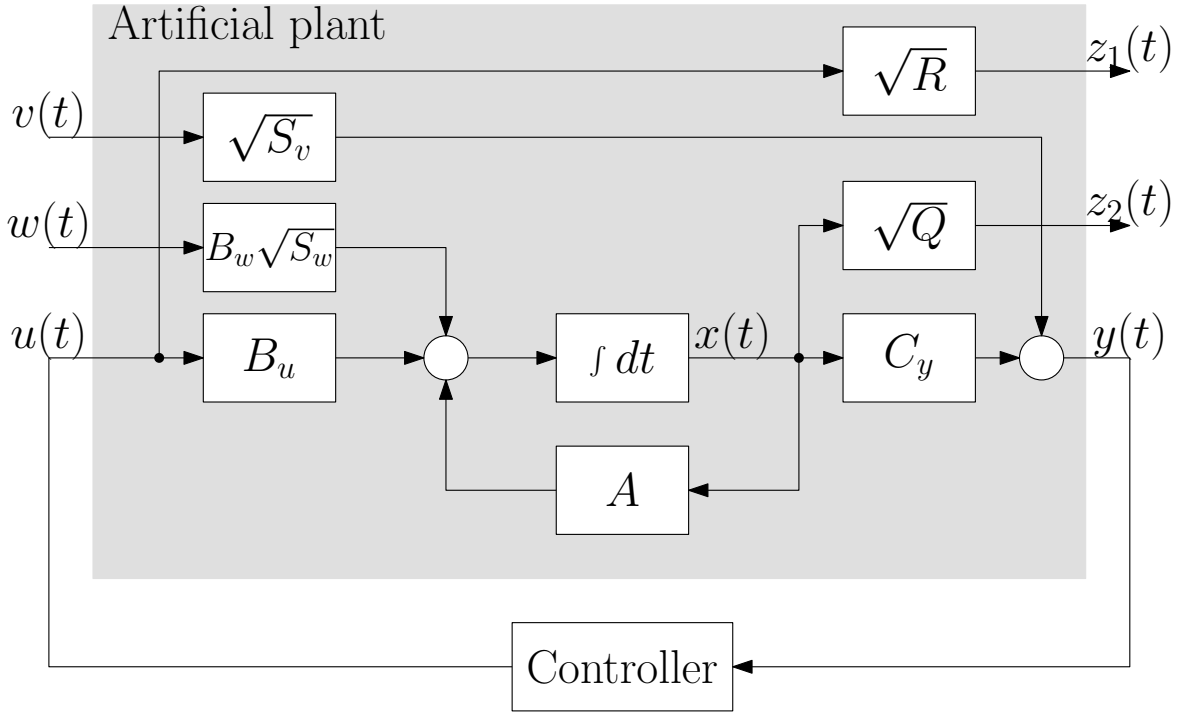

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  - measured outputs: $y(t)$



$\mathcal{H}_2$-optimal control in full generality



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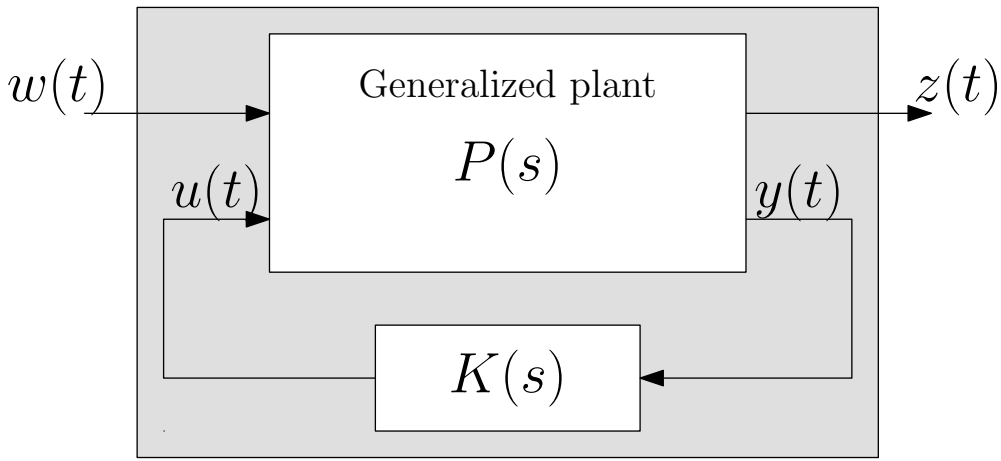
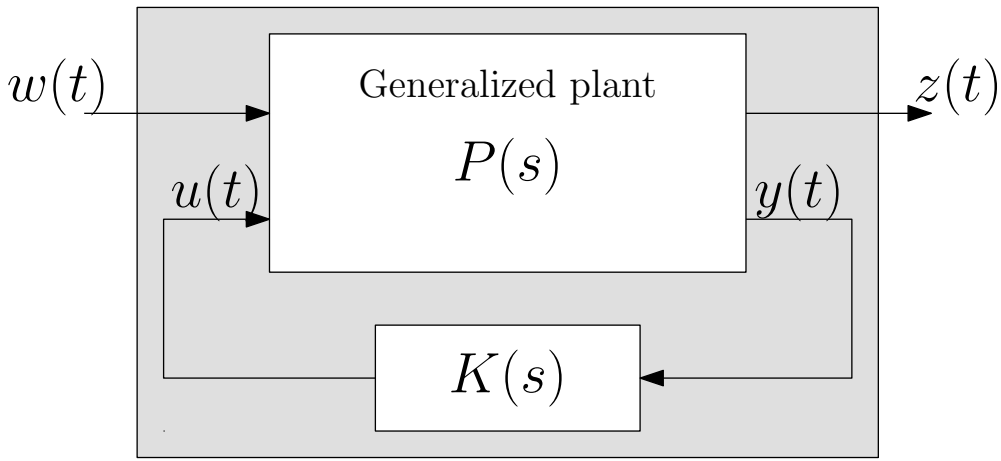


$\mathcal{H}_2$-optimal control in full generality



$$\underset{K(s)\text{ stabilizing}}{\text{minimize}} \quad \|N(P,K)\|_2$$

To conclude the LQ story



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1. design usable controllers by minimizing the integral of some functions of the states and the inputs
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How about using other system norms?

