# Optimization with inequalities: KKT conditions

Graduate course on Optimal and Robust Control (spring'22)

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February 23, 2022

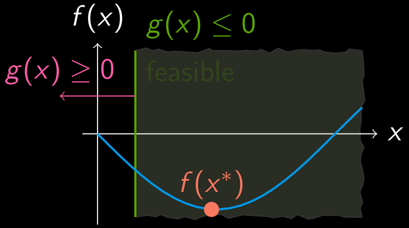
$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}}\, f(\boldsymbol{x})$$

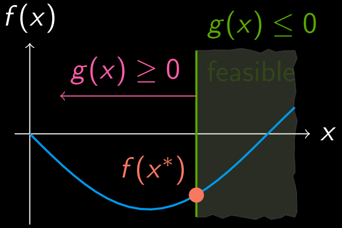
$$\text{subject to } \mathbf{g}(\boldsymbol{x}) \leq \mathbf{0}$$

Eventually combined with equality constraints

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}}\ & f(\boldsymbol{x}) \\ \text{subject to } \mathbf{g}(\boldsymbol{x}) &\leq \mathbf{0} \\ \mathbf{h}(\boldsymbol{x}) &= \mathbf{0} \end{aligned}$$

$\text{minimize}_{x \in \mathbb{R}} f(x)$ subject to $g(x) \leq 0$




$$\mathrm{d}f = f'(x)\mathrm{d}x \geq 0$$

$$\boxed{f'(x) = 0}$$

$$\mathrm{d}f = f'(x)\mathrm{d}x \geq 0$$

$$\mathrm{d}g = g'(x)\mathrm{d}x \leq 0$$

$$\boxed{\operatorname{sign} f'(x) = -\operatorname{sign} g'(x)}$$

$$\boxed{f'(x) + \mu g'(x) = 0, \quad \mu \geq 0}$$

$$\boxed{\begin{aligned} f'(x) + \mu g'(x) &= 0 \\ \mu &\geq 0 \\ g(x) &\leq 0 \end{aligned}}$$

But more can be said

- Either $g(x) = 0$,
- or $g(x) < 0 \implies \mu = 0$.

As a result: complementary slackness condition

$$\boxed{\mu g(x) = 0}$$

# KKT conditions – necessary conditions of optimality

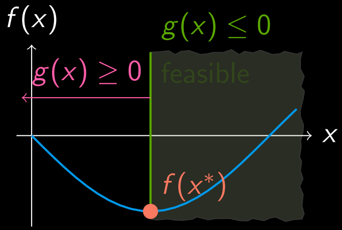
Karush-Kuhn-Tucker

| | |
|---|---|
| $f'(x) + \mu g'(x) = 0$ | stationarity condition |
| $\mu \geq 0$ | dual feasibility |
| $g(x) \leq 0$ | primal feasibility |
| $\mu g(x) = 0$ | complementary slackness |

# Complementary slackness is not necessarily strict

Generally it is not excluded that both $g(x)$ and $\mu$ are zero. But is strict for LP.

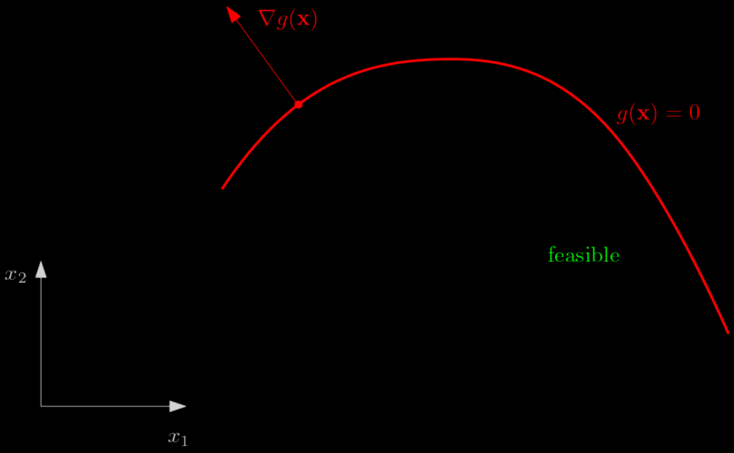
Example:




$$g(x) = 0,\ f'(x) = 0,\ g'(x) \neq 0\ (g(x) = a - x \Rightarrow g'(x) = 1).$$

# Vector case (several variables): single constraint

$\boldsymbol{x} \in \mathbb{R}^n$, say $n = 2$



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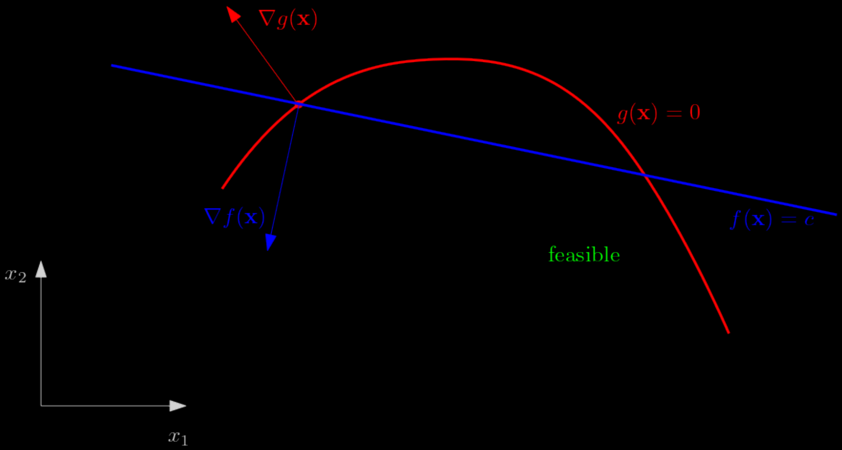

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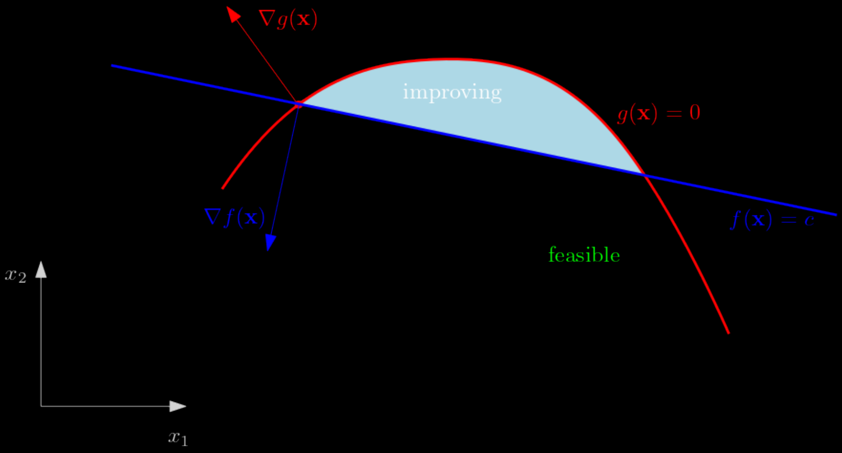
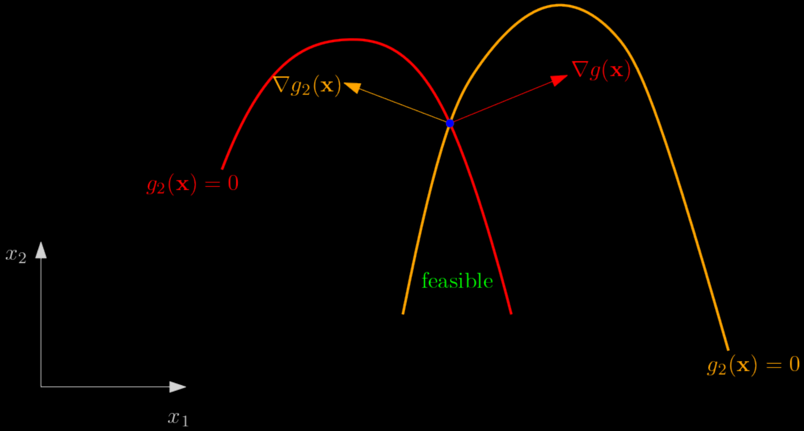


# Vector case (several variables): two (and more) constraints



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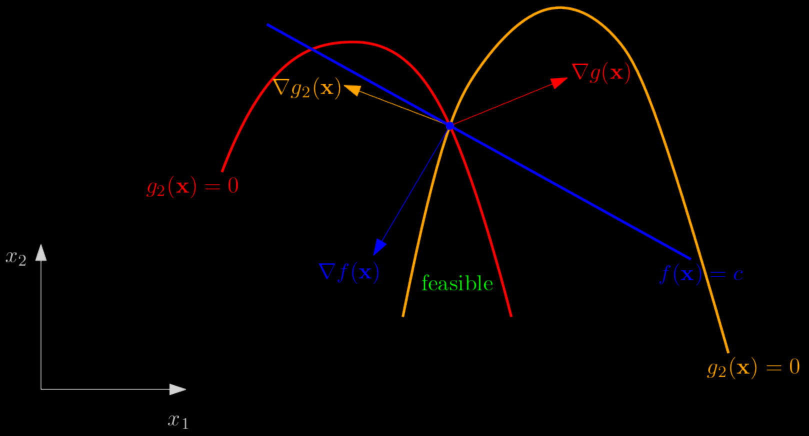
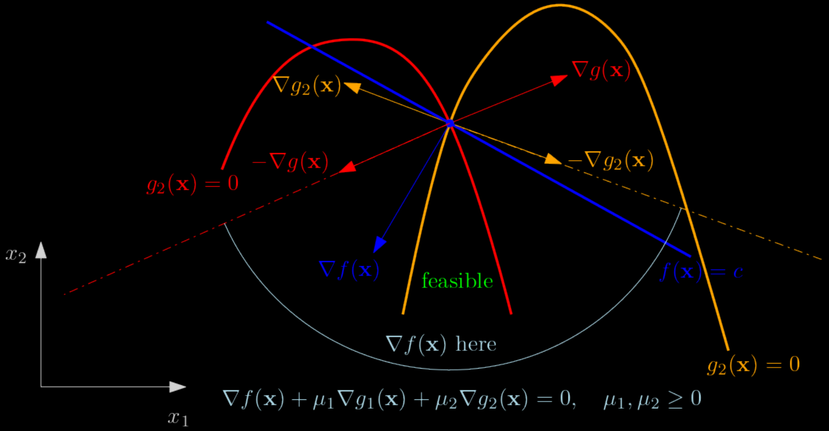


# Vector case (several variables): two (and more) constraints



$$\nabla f(\boldsymbol{x}) + \sum_{i}^{m} \mu_i \nabla g_i(\boldsymbol{x}) = 0$$

$$\mu_i \geq 0, \quad i = 1, \ldots, m$$

$$g_i(\boldsymbol{x}) \leq 0, \quad i = 1, \ldots, m$$

$$\mu_i g_i(\boldsymbol{x}) = 0, \quad i = 1, \ldots, m$$

# Constraint qualification needed for active constraints

Inequality constrants active at $\boldsymbol{x}$ are those satisfied with equality

$$g_i(\boldsymbol{x}) = 0.$$

Active constraints must satisfy some regularity conditions (also called constraint qualification) for the KKT conditions to be necessary at a given point.
One of several possible constraint qualifications for KKT conditions is Linear Independence Constraint Qualification (LICQ) (= linear independence of gradients of active constraints (as in the equality case).

# Constraint qualification for convex optimization

For convex problems ($f$ convex, $g_i$s convex, $h_j$s affine), verify the existence of Slater's point:
$\boldsymbol{x}$ such that inequality constraints are satisfied strictly (also called strictly feasible point):

$$g_i(x) < 0 \quad i = 1, \ldots, m$$

# KKT conditions – necessary conditions of optimality for inequalities

Assume constraint qualification (existence of Slater's point in the convex case). The necessary conditions are

$$\boxed{\begin{aligned} \nabla f(\boldsymbol{x}) + \sum_{i}^{m} \mu_i \nabla g_i(\boldsymbol{x}) = 0 & \\ \mu_i \geq 0, \quad & i = 1, \ldots, m \\ g_i(\boldsymbol{x}) \leq 0, \quad & i = 1, \ldots, m \\ \mu_i g_i(\boldsymbol{x}) = 0, \quad & i = 1, \ldots, m \end{aligned}}$$

# KKT conditions in vector format

$$\boxed{\begin{aligned} \nabla f(\boldsymbol{x}) + \nabla \mathbf{g}(\boldsymbol{x})\boldsymbol{\mu} &= \mathbf{0} \\ \boldsymbol{\mu} &\geq \mathbf{0} \\ \mathbf{g}(\boldsymbol{x}) &\leq \mathbf{0} \\ \boldsymbol{\mu}^\top \mathbf{g}(\boldsymbol{x}) &= \mathbf{0} \end{aligned}}$$

where

$$\nabla \mathbf{g}(\boldsymbol{x}) = \begin{bmatrix} \nabla g_1(\boldsymbol{x}) & \nabla g_2(\boldsymbol{x}) & \dots & \nabla g_m(\boldsymbol{x}) \end{bmatrix}$$

is a Jacobian (matrix) for the constraints.

# KKT conditions as necessary cond's for inequality and equality constraints

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}}\ & f(\boldsymbol{x}) \\ \text{subject to } & \mathbf{g}(\boldsymbol{x}) \leq \mathbf{0} \\ & \mathbf{h}(\boldsymbol{x}) = \mathbf{0} \end{aligned}$$

Assume constraint qualification (Slater's point in convex case)

$$\nabla f(\boldsymbol{x}) + \sum_{i}^{m} \mu_i \nabla g_i(\boldsymbol{x}) + \sum_{i}^{m} \lambda_i \nabla h_i(\boldsymbol{x}) = 0$$

$$\begin{aligned} \mu_i \geq 0, &\quad i = 1, \ldots, m \\ g_i(\boldsymbol{x}) \leq 0, &\quad i = 1, \ldots, m \\ h_i(\boldsymbol{x}) = 0, &\quad i = 1, \ldots, p \\ \mu_i g_i(\boldsymbol{x}) = 0, &\quad i = 1, \ldots, m \end{aligned}$$

# KKT conditions also sufficient for convex case

Generally some second-order necessary conditions must be developed and satisfied, but for a convex problem just satisfaction of KKT conditions suffices.
But don't forget the assumption of existence of Slater's point.