① a)

$$\nabla \times F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\sin y + e^{3z} & y + x^2\cos y & 3xe^{3z} - 1 \end{vmatrix}$$

$$= \left(0-0,\ 3e^{3z}-3e^{3z},\ 2x\cos y - 2x\cos y\right) = (0,0,0) \text{ na } \mathbb{R}^2.$$

$\Rightarrow$ F je potenciálové na $\mathbb{R}^2$.

b) $\frac{\partial f}{\partial x} = 2x\sin y + e^{3z}$ (1)

$\frac{\partial f}{\partial y} = y + x^2\cos y$ (2)

$\frac{\partial f}{\partial z} = 3xe^{3z} - 1$ (3)

(1) $\Rightarrow f(x,y,z) = x^2\sin y + xe^{3z} + C_1(y,z)$

Dosazením do (2) máme : $\cancel{x^2\cos y} + \frac{\partial C_1}{\partial y} = y + \cancel{x^2\cos y}$

$\Rightarrow \frac{\partial C_1}{\partial y} = y \Rightarrow C_1(y,z) = \frac{y^2}{2} + C_2(z).$

Odtud $f(x,y,z) = x^2\sin y + xe^{3z} + \frac{y^2}{2} + C_2(z).$

Dosazením do (3) dostaneme : $3xe^{3z} + C_2'(z) = 3xe^{3z} - 1$

$\Rightarrow C_2'(z) = -1 \Rightarrow C_2(z) = -z + K$, $K \in \mathbb{R}$.

Protože $0 = f(1,0,0) = 1 + C_2(0) = 1 + K$, je $K = -1$.

Tedy

$$f(x,y,z) = x^2\sin y + xe^{3z} + \frac{y^2}{2} - z - 1 .$$

② $\frac{\partial f}{\partial x} = 3x^2 + y^2 - 12 = 0$

$\frac{\partial f}{\partial y} = 2xy + 2y = 0 \quad \ldots \quad 2y(x+1) = 0 \begin{cases} y = 0 \\ x = -1 \end{cases}$

- $y = 0 \Rightarrow 3x^2 = 12$
  $x = \pm 2$
- $x = -1 \Rightarrow y^2 = 9$
  $y = \pm 3$

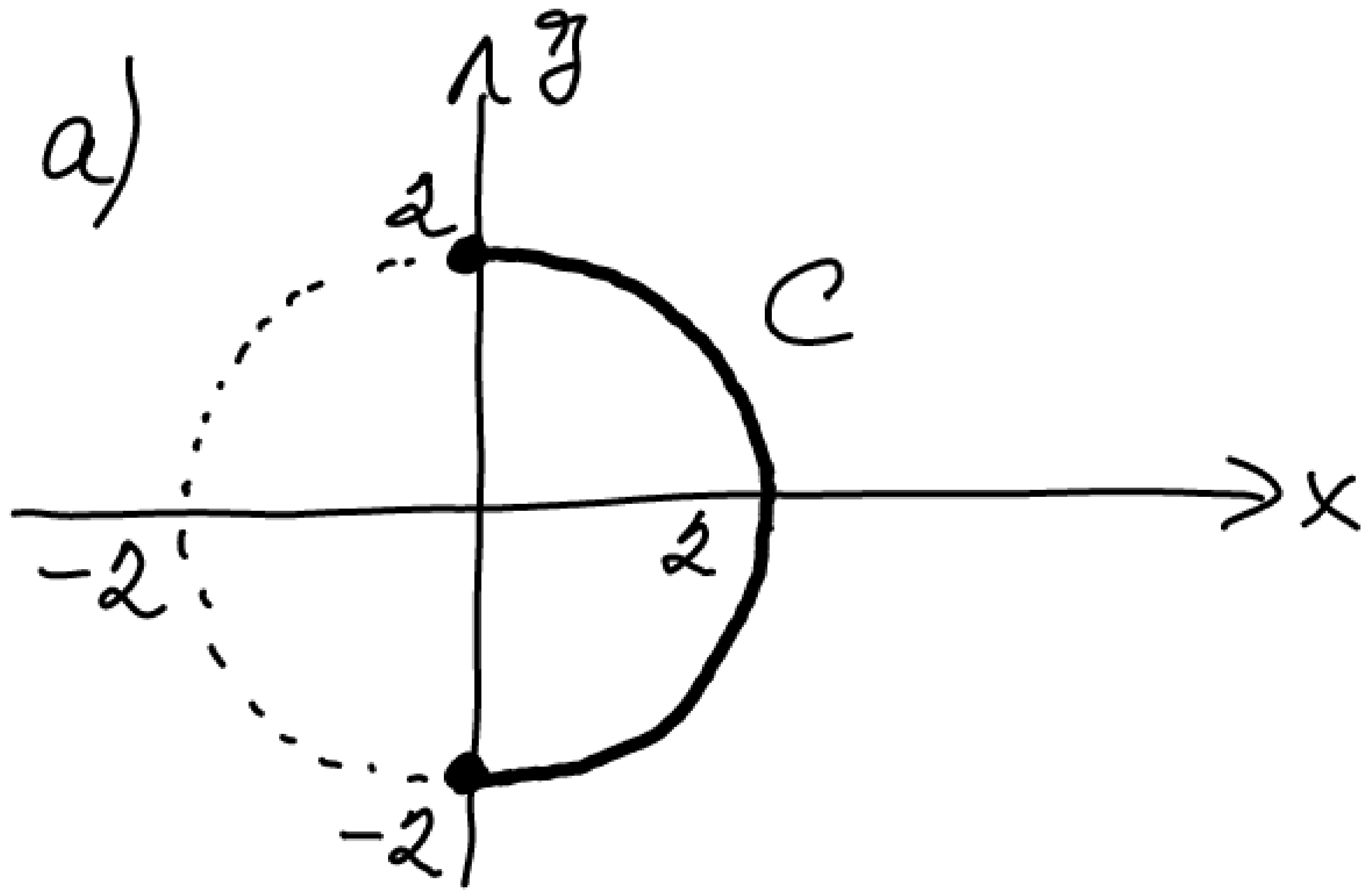
Stacionární body jsou: $A = (2, 0)$, $B = (-2, 0)$, $C = (-1, 3)$, $D = (-1, -3)$

Hessova matice:

$$H_f(x, y) = \begin{pmatrix} 6x & 2y \\ 2y & 2x+2 \end{pmatrix}$$

- $H_f(A) = \begin{pmatrix} 12 & 0 \\ 0 & 6 \end{pmatrix}$ je pozit. def. $\Rightarrow$ A je bod lok. min.
- $H_f(B) = \begin{pmatrix} -12 & 0 \\ 0 & -2 \end{pmatrix}$ je negat. def. $\Rightarrow$ B je bod lok. max.
- $H_f(C) = \begin{pmatrix} -6 & 6 \\ 6 & 0 \end{pmatrix} \ldots \det H_f(C) = -36 < 0 \Rightarrow$
  $\Rightarrow H_f(C)$ je indefinitní $\Rightarrow$ C je sedlový bod
- $H_f(D) = \begin{pmatrix} -6 & -6 \\ -6 & 0 \end{pmatrix} \ldots \det H_f(D) = -36 < 0 \Rightarrow$
  $\Rightarrow H_f(D)$ je indefinitní $\Rightarrow$ D je sedlový bod.

3 a)



$C \ldots \varphi(t) = (2\cos t, 2\sin t)$

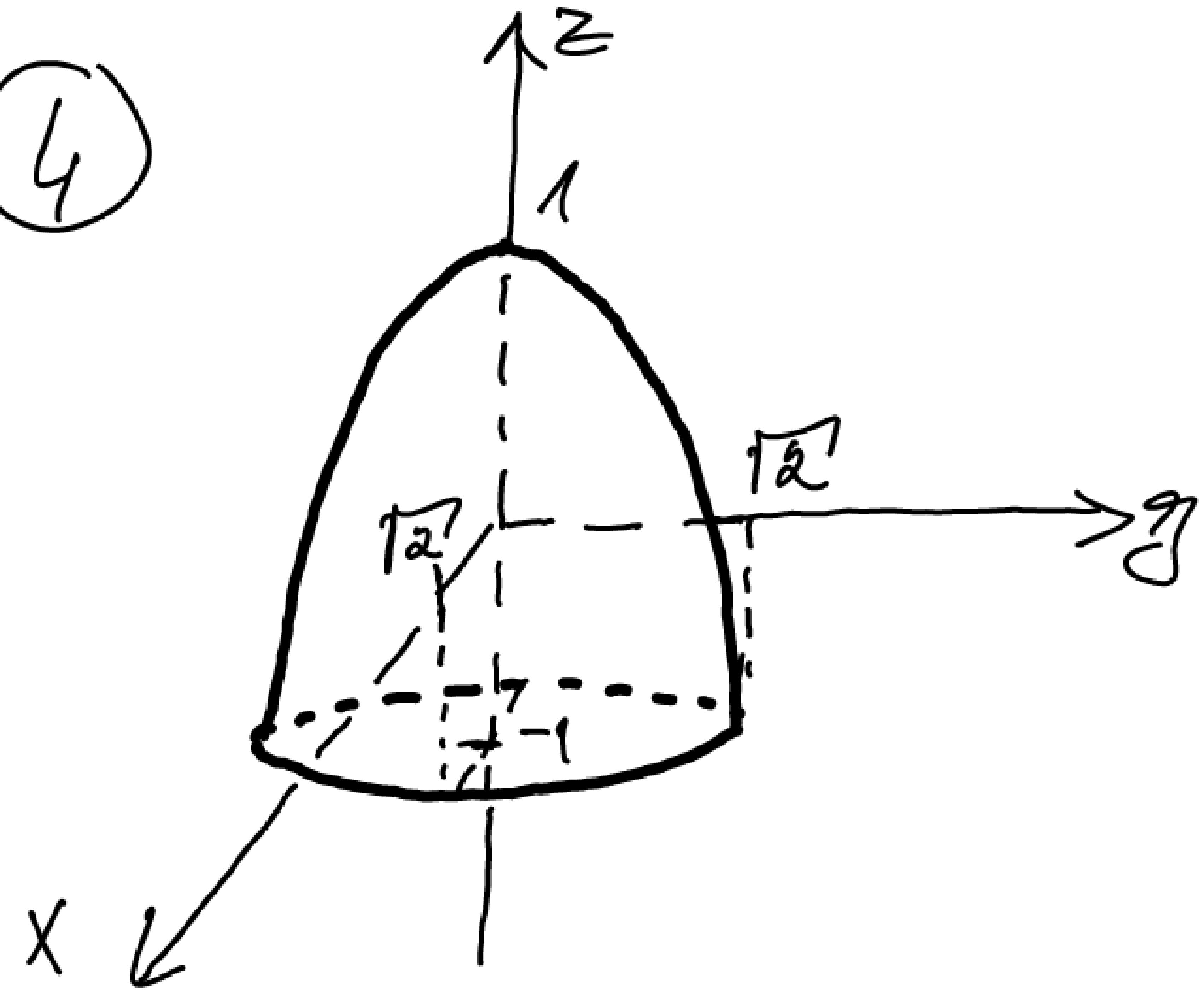
$t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b) $$\int_C x^2 + y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos^2 t + 2\sin t) \overbrace{\sqrt{4\cos^2 t + 4\sin^2 t}}^{=2}\, dt$$

$$= 8 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt + 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t \, dt$$

$$= 16 \int_0^{\frac{\pi}{2}} \frac{1 + \cos(2t)}{2} \, dt = 8 \frac{\pi}{2} = 4\pi$$

4



$$M = \{(x,y,z) \mid -1 \le z \le 1 - x^2 - y^2\}$$

$$\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + 2y + 2z$$

$$\int_S F = \int_M \nabla \cdot F = \int_M 1 + 2y + 2z$$

$$= \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{\sqrt{1-z}} (1 + 2r\sin\varphi + 2z)\, r \, dr \, d\varphi \, dz$$

$$= 2\pi \int_{-1}^{1} \int_{0}^{\sqrt{1-z}} r(1+2z) \, dr \, dz = 2\pi \int_{-1}^{1} \frac{1-z}{2}(1+2z) \, dz$$

$$= \pi \int_{-1}^{1} 1 + z - 2z^2 \, dz = \pi\left(2 + 0 - \frac{2}{3}2\right)$$

$$= \frac{2\pi}{3}$$

5 a) $\lim_{k\to\infty} \sqrt[k]{\left|\frac{(x-1)^k}{3^k k}\right|} = \lim_{k\to\infty} \frac{|x-1|}{3\sqrt[k]{k}} = \frac{|x-1|}{3} < 1$

$\Leftrightarrow |x-1| < 3$ ... polom. konv. je $R = 3$.

Řada diverguje v bodě -3, neboť $|-3-1| = 4 > 3$.

b) $\sum_{k=1}^{\infty} \frac{(x-1)^k}{3^k k} \overset{\text{Der.}}{\rightsquigarrow} \sum_{k=1}^{\infty} \frac{(x-1)^{k-1}}{3^k} = \frac{1}{3} \sum_{k=0}^{\infty} \frac{(x-1)^k}{3^k}$

$= \frac{1}{3} \frac{1}{1 - \frac{x-1}{3}} = \frac{1}{4-x} \quad , x \in (-2, 4)$

$\Rightarrow \sum_{k=1}^{\infty} \frac{(x-1)^k}{3^k k} = \int \frac{1}{4-x} dx = -\ln(4-x) + C$.

$x = 1 \ldots \quad 0 = -\ln(3) + C \Leftrightarrow C = \ln 3$.

Tedy $\sum_{k=1}^{\infty} \frac{(x-1)^k}{3^k k} = -\ln(4-x) + \ln 3, x \in (-2, 4)$.