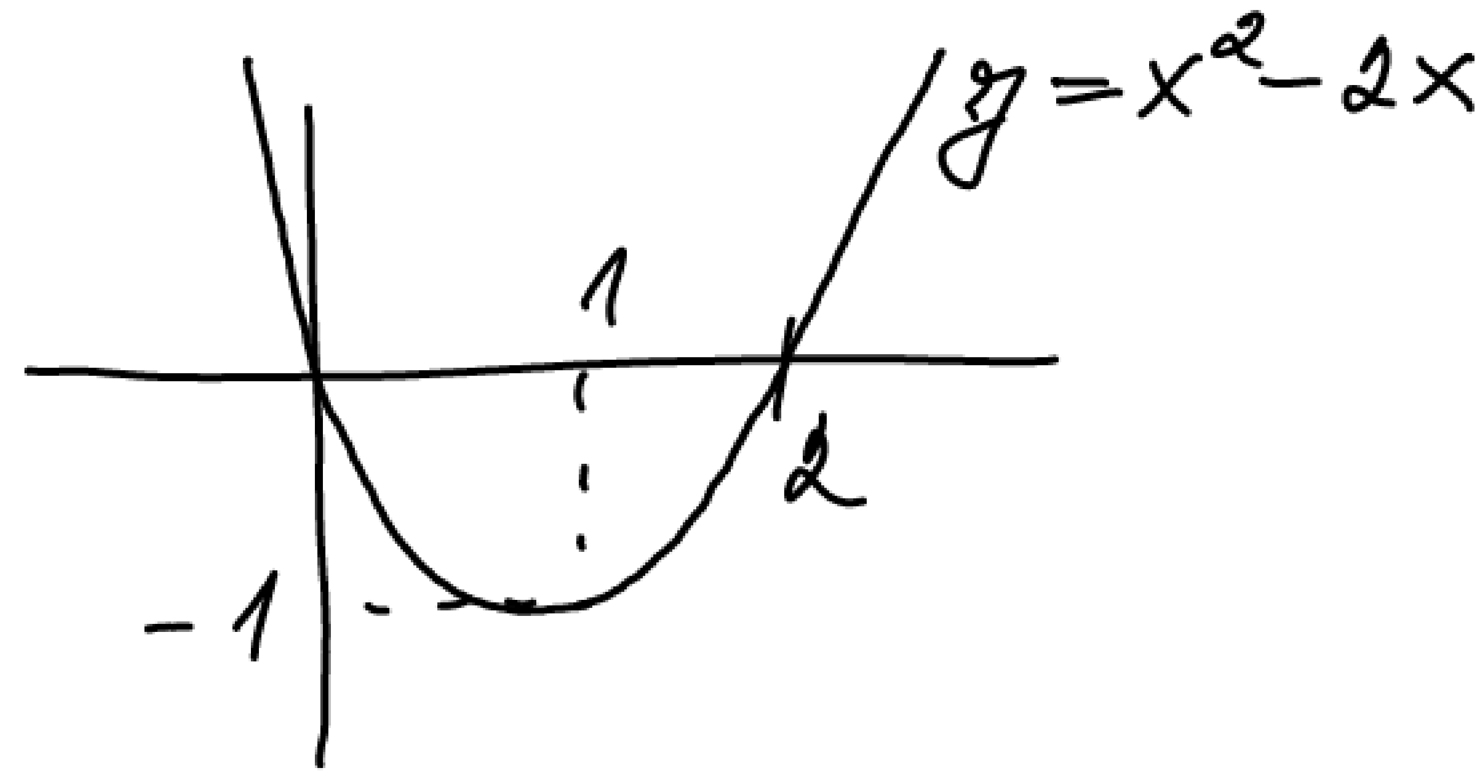
① a) $f(x,y)=1 \iff 2x+y+1 = x^2+1 \iff y = x^2-2x$

Tedy $\text{lev}(f;1) = \{(x, x^2-2x) \mid x \in \mathbb{R}\}$




b) $g'(-2) = \frac{\partial f}{\partial x}(1,1)\varphi_1'(-2) + \frac{\partial f}{\partial y}(1,1)\varphi_2'(-2)$

$$\frac{\partial f}{\partial x}(1,1) = \frac{2(x^2+1)-(2x+y+1)2x}{(x^2+1)^2}\Bigg|_{\substack{x=1\\y=1}} = \frac{4-8}{4} = -1$$

$$\frac{\partial f}{\partial y}(1,1) = \frac{1}{x^2+1}\Bigg|_{x=1} = \frac{1}{2}$$

$$\Rightarrow g'(-2) = -1 + 2\cdot\frac{1}{2} = 0$$

② $4x^3 - y = 0 \qquad (1)$

$2y - x = 0 \qquad (2) \iff y = \frac{1}{2}x$

Dosazení do (1): $\frac{1}{2}x(8x^2 - 1) = 0$

$x = 0$ nebo $x = \pm\frac{1}{2\sqrt{2}}$

Stacionární body proto jsou

$(0,0)$, $\left(\frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}\right)$, $\left(-\frac{1}{2\sqrt{2}}, -\frac{1}{4\sqrt{2}}\right)$.

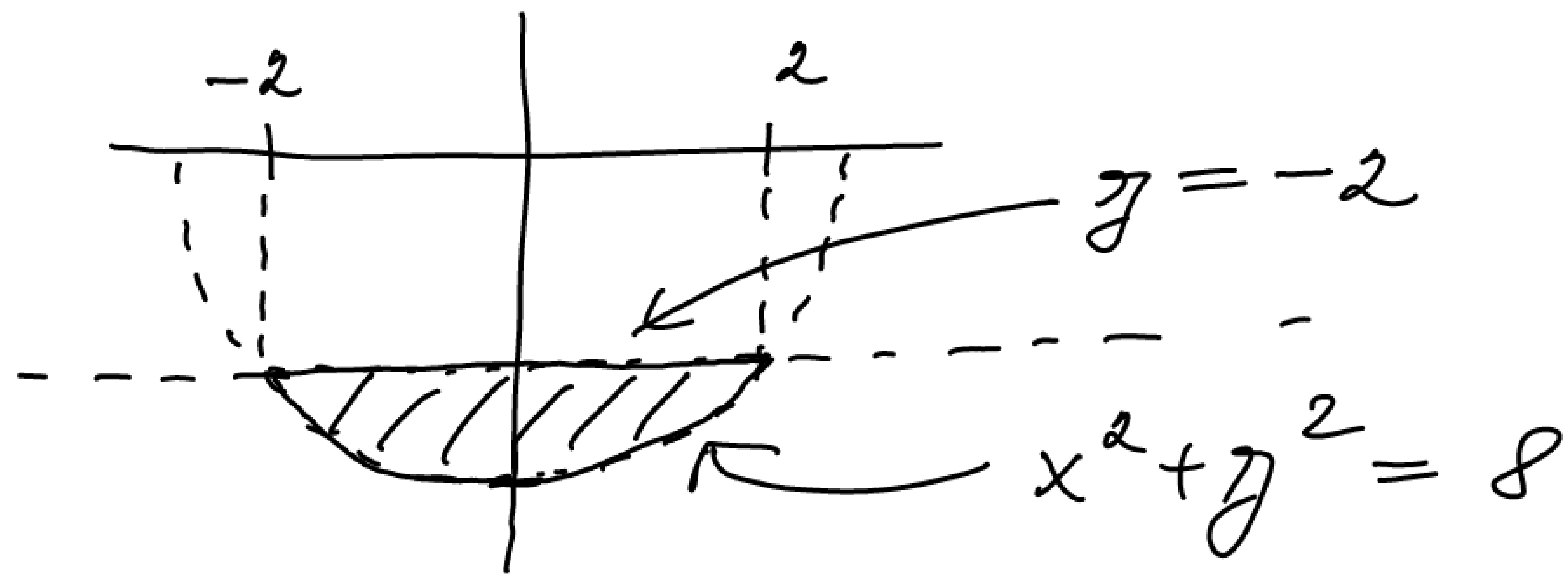
$$H_f(x,y) = \begin{pmatrix} 12x^2 & -1 \\ -1 & 2 \end{pmatrix}$$

- $H_f(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$ je indefinitní (neboť $\det H_f(0,0) < 0$) $\Rightarrow (0,0)$ je sedlový bod.
- $H_f\left(\frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}\right) = \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & 2 \end{pmatrix}$ je pozitivně definitní (neboť $\frac{3}{2} > 0$ a $\det H_f\left(\frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}\right) = 2 > 0$) $\Rightarrow$

$\Rightarrow \left(\frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}\right)$ je bod lokálního minima.

- $H_f\left(-\frac{1}{2\sqrt{2}}, -\frac{1}{4\sqrt{2}}\right) = \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow \left(\frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}\right)$ je bod lokálního minima.

(3)




$y=-2 \ldots \quad r\sin\varphi = -2$, tj. $r = \frac{-2}{\sin\varphi}$

Tedy $\varphi \in \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right]$ a $r \in \left[-\frac{2}{\sin\varphi}, 2\sqrt{2}\right]$

Proto

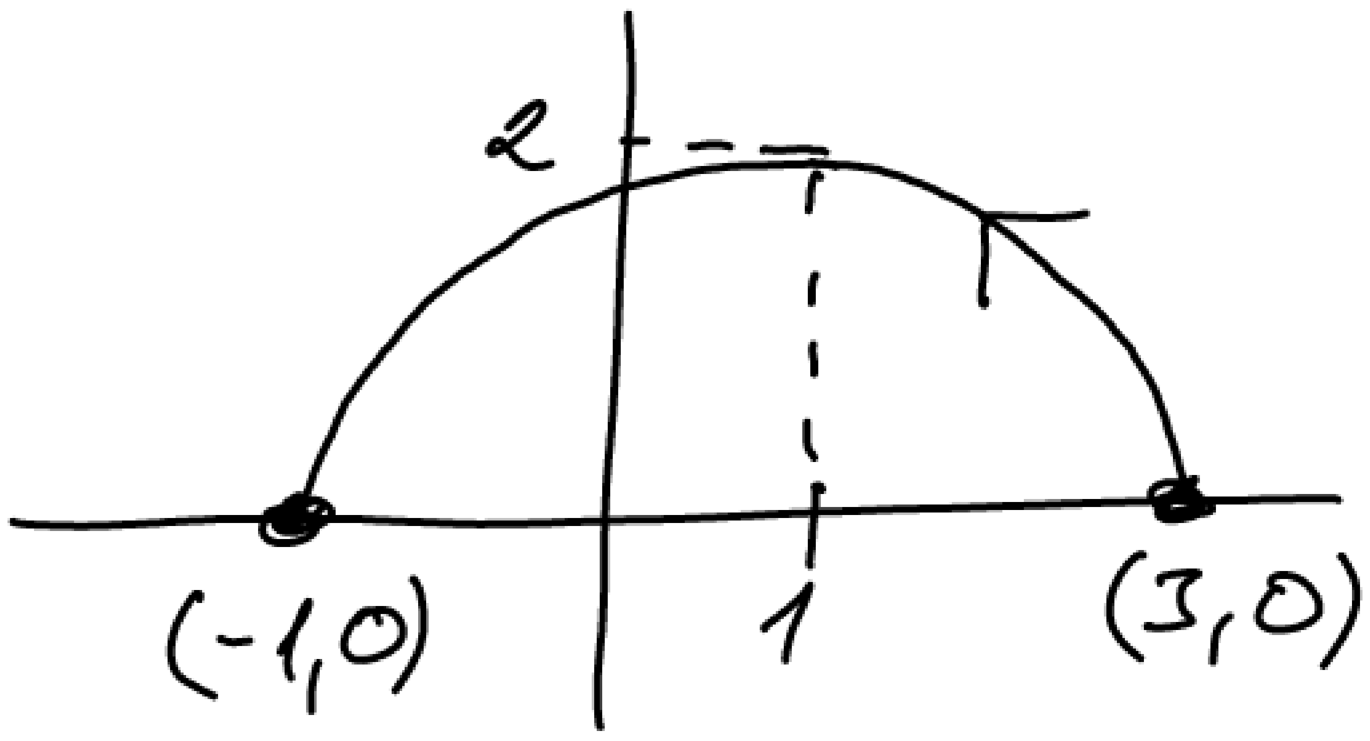
$$\int_{-2}^{2}\int_{-\sqrt{8-x^2}}^{-2} \frac{y}{x^2+y^2}\,dy\,dx = \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}}\int_{-\frac{2}{\sin\varphi}}^{2\sqrt{2}} \frac{r\sin\varphi}{r^2}\, r\,dr\,d\varphi$$

$$= \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} \left(2\sqrt{2} + \frac{2}{\sin\varphi}\right)\sin\varphi\, d\varphi = 2\sqrt{2}\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} \sin\varphi\, d\varphi +$$

$$+ 2\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} d\varphi = 2\sqrt{2}\left[-\cos\varphi\right]_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} + 2\,\frac{\pi}{2}$$

$$= -2\sqrt{2}\left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right] + \pi = \pi - 4$$

(4)




$$\varphi(t) = (1+2\cos t, 2\sin t),$$
$$t \in [0, \pi]$$
$$\varphi'(t) = (-2\sin t, 2\cos t)$$

$$\Rightarrow \int_C F(x,y) \cdot ds = \int_0^{\pi} (2\cos t, (1+2\cos t)2\sin t) \cdot (-2\sin t, 2\cos t)\, dt$$

$$= \int_0^{\pi} \cancel{-4\sin t\cos t + 4\sin t\cos t} + 8\sin t\cos^2 t\, dt$$

$$= 8\left[-\frac{\cos^3 t}{3}\right]_0^{\pi} = -\frac{8}{3}(-1-1) = \frac{16}{3}$$