Směrová a parciální derivace

Zadání

1. Z definice nalezněte $\nabla_{\boldsymbol{v}} f(\boldsymbol{a})$, jestliže
   (a) $f(x,y,z) = x^2y^2(2z+1)^2$, $\boldsymbol{a} = (1,-1,1)$, $\boldsymbol{v} = (0,1,3)$;
   (b) $f(x,y) = \frac{xy}{x+y}$, $\boldsymbol{a} = (2,-1)$, $\boldsymbol{v} = (4,1)$;
   (c) $f(x,y) = e^{3x}\cos(x-y)$, $\boldsymbol{a} = (0,\frac{\pi}{2})$, $\boldsymbol{v} = (1,2)$.

2. Ať $f(x,y) = \sqrt{|xy|}$. Nalezněte všechny vektory $\boldsymbol{v}$ tak, aby směrová derivace $\nabla_{\boldsymbol{v}} f(0,0)$ existovala a tuto derivaci určete.

3. Je dána funkce
   $$f(x,y) = \begin{cases} \frac{x^2y^3}{x^4+y^4}, & \text{je-li } (x,y) \neq (0,0); \\ 0, & \text{je-li } (x,y) = (0,0). \end{cases}$$
   Nalezněte všechny vektory $\boldsymbol{v}$ tak, aby směrová derivace $\nabla_{\boldsymbol{v}} f(0,0)$ existovala a tuto derivaci určete. Je zobrazení $\boldsymbol{v} \mapsto \nabla_{\boldsymbol{v}} f(0,0)$ lineární? Je funkce f spojitá?

4. Nalezněte všechny parciální derivace (1. řádu) funkce f v bodě $\boldsymbol{a}$, jestliže
   (a) $f(x,y) = x^2 - 3x^2y + 5y^3$, $\boldsymbol{a} = (-2,1)$;
   (b) $f(x,y) = \sqrt{3x^2+y}$, $\boldsymbol{a} = (1,2)$;

5. Nalezněte všechny parciální derivace (1. řádu) funkce f ve všech bodech, kde existují, jestliže
   (a) $f(x,y) = x\sin(xy)$;
   (b) $f(x,y,z) = \sqrt{x^2+y^2\cos^2 z+1}$;
   (c) $f(t,x,y,z) = x^2y\cos\frac{z}{t}$;
   (d) $f(x,y) = \frac{2x-y}{y-3x}$;
   (e) $f(x,y) = x^y$;
   (f) $f(x,y) = \sqrt{x^2+y^2}$;
   (g) $f(x,y,z) = xy\ln(xz)$;
   (h) $f(x,y) = \int_y^x \cos\left(e^t\right)\,\mathrm{d}t$.

6. Vypočtěte všechny parciální derivace (1. řádu) funkce f a určete všechny body, ve kterých jsou spojité, jestliže
   $$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0); \end{cases}$$

7. Ukažte, že $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$, jestliže
   $$f(x,y) = \begin{cases} xy, & |x| \geq |y|, \\ 0, & |x| < |y|. \end{cases}$$

8. Je dána funkce $f(x,y) = \frac{y}{2x+3y}$. Přímým výpočtem ověřte, že $\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$.

9. Vypočtěte všechny parciální derivace 1. a 2. řádu funkce f ve všech bodech, kde existují, jestliže

   (a) $f(x,y) = e^{xy} \sin y$;

   (b) $f(x,y) = \operatorname{arctg} \frac{y}{x}$;

   (c) $f(x,y) = x^3y^2 - 2xy + \cos(x^2 - y^2)$;

   (d) $f(x,y) = \sqrt{1 + xy^2}$.

10. Je dána funkce $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. Ukažte, že $\Delta f = 0$ na $\mathbb{R}^3 \setminus \{0\}$.

11. Je dána funkce $f(t,x) = \cos(x - ct) + \sin(y + ct)$, kde $c > 0$. Ukažte, že $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$.

12. Ukažte, že $f(t,x) = \frac{1}{\sqrt{\alpha t}} e^{-\frac{x^2}{4\alpha t}}$, kde $\alpha > 0$, vyhovuje rovnici $\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$ na $(0,\infty) \times \mathbb{R}$.

13. Ukažte, že $f(t,x) = e^{-t} \cos\left(\frac{x}{\alpha}\right)$, kde $\alpha > 0$, vyhovuje rovnici $\frac{\partial f}{\partial t} = \alpha^2 \frac{\partial^2 f}{\partial x^2}$ na $\mathbb{R}^2$.

Výsledky

1. Nalezněte $\nabla_{\boldsymbol{v}} f(a)$. jestliže
   (a) $\nabla_{\boldsymbol{v}} f(a) = 18$;
   (b) $\nabla_{\boldsymbol{v}} f(a) = 8$;
   (c) $\nabla_{\boldsymbol{v}} f(a) = -1$.

2. $\nabla_{\boldsymbol{v}} f(0,0)$ existuje právě tehdy, když alespoň jedna komponenta vektoru $\boldsymbol{v}$ je nulová. V tomto případě je $\nabla_{\boldsymbol{v}} f(0,0) = 0$.

3. $\nabla_{(0,0)} f(0,0) = 0$, $\nabla_{(v_1,v_2)} f(0,0) = \frac{v_1^2 v_2^3}{v_1^4 + v_2^4}$ pro každé $(v_1, v_2) \neq (0,0)$. Zobrazení $\boldsymbol{v} \mapsto \nabla_{\boldsymbol{v}} f(0,0)$ není lineární. Funkce f je spojitá.

4. (a) $\frac{\partial f}{\partial x}(a) = 8$, $\frac{\partial f}{\partial y}(a) = 3$;
   (b) $\frac{\partial f}{\partial x}(a) = \frac{3}{\sqrt{5}}$, $\frac{\partial f}{\partial x}(a) = \frac{1}{2\sqrt{5}}$;

5. (a) Pro $(x,y) \in \mathbb{R}^2$ je $\frac{\partial f}{\partial x}(x,y) = \sin(xy) + xy\cos(xy)$, $\frac{\partial f}{\partial y}(x,y) = x^2\cos(xy)$.
   (b) Pro $(x,y,z) \in \mathbb{R}^3$ je
   $$\frac{\partial f}{\partial x}(x,y,z) = \frac{x}{\sqrt{x^2 + y^2\cos^2 z + 1}}, \quad \frac{\partial f}{\partial y}(x,y,z) = \frac{y\cos z}{\sqrt{x^2 + y^2\cos^2 z + 1}},$$
   $$\frac{\partial f}{\partial z}(x,y,z) = -\frac{y^2 \sin z \cos z}{\sqrt{x^2 + y^2\cos^2 z + 1}}.$$
   (c) Pro $t \neq 0$ a $x, y, z \in \mathbb{R}$ je
   $$\frac{\partial f}{\partial t}(t,x,y,z) = \frac{x^2yz\sin\frac{z}{t}}{t^2}, \quad \frac{\partial f}{\partial x}(t,x,y,z) = 2xy\cos\frac{z}{t},$$
   $$\frac{\partial f}{\partial y}(t,x,y,z) = x^2\cos\frac{z}{t}, \quad \frac{\partial f}{\partial z}(t,x,y,z) = -\frac{x^2y\sin\frac{z}{t}}{t}.$$
   (d) Pro $y \neq 3x$ je $\frac{\partial f}{\partial x}(x,y) = -\frac{y}{(y-3x)^2}$, $\frac{\partial f}{\partial y}(x,y) = \frac{x}{(y-3x)^2}$.
   (e) Pro $x > 0$ a $y \in \mathbb{R}$ je $\frac{\partial f}{\partial x}(x,y) = yx^{y-1}$, $\frac{\partial f}{\partial y}(x,y) = x^y \ln x$.
   (f) Pro $(x,y) \neq (0,0)$ je $\frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}}$.
   (g) Pro $xz > 0$ a $y \in \mathbb{R}$ je
   $$\frac{\partial f}{\partial x}(x,y,z) = y(\ln(xz) + 1), \quad \frac{\partial f}{\partial y}(x,y,z) = x\ln(xz),$$
   $$\frac{\partial f}{\partial z}(x,y,z) = \frac{xy}{z}.$$
   (h) Pro $(x,y) \in \mathbb{R}^2$ je $\frac{\partial f}{\partial x}(x,y) = \cos(e^x)$, $\frac{\partial f}{\partial y}(x,y) = -\cos(e^y)$.

6. Pro $(x,y) \neq (0,0)$ je $\frac{\partial f}{\partial x}(x,y) = \frac{y^3 - x^2y}{(x^2+y^2)^2}$ a $\frac{\partial f}{\partial y}(x,y) = \frac{x^3 - xy^2}{(x^2+y^2)^2}$. Navíc $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Parciální derivace nejsou spojité jen v bodě $(0,0)$.

8. $\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{6y-4x}{(2x+3y)^3} = \frac{\partial^2 f}{\partial y \partial x}(x,y)$.

9. (a) Je-li $(x,y) \in \mathbb{R}^2$, pak

$$\frac{\partial f}{\partial x}(x,y) = ye^{xy}\sin y, \quad \frac{\partial f}{\partial y}(x,y) = e^{xy}\left(x\sin y + \cos y\right),$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = y^2 e^{xy}\sin y, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = e^{xy}\left[(x^2-1)\sin y + 2x\cos y\right]$$
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = e^{xy}\left(xy\sin y + \sin y + y\cos y\right).$$

(b) Je-li $x \neq 0$ a $y \in \mathbb{R}$, pak

$$\frac{\partial f}{\partial x}(x,y) = \frac{-y}{x^2+y^2}, \quad \frac{\partial f}{\partial y}(x,y) = \frac{x}{x^2+y^2},$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{-2xy}{(x^2+y^2)^2}$$
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

(c) Je-li $(x,y) \in \mathbb{R}$, pak

$$\frac{\partial f}{\partial x}(x,y) = y(3x^2y-2) - 2x\sin(x^2-y^2)$$
$$\frac{\partial f}{\partial y}(x,y) = 2\left[x^3y + y\sin(x^2-y^2) - x\right],$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = -2\left[\sin(x^2-y^2) + 2x^2\cos(x^2-y^2) - 3xy^2\right],$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = 2\left[x^3 + \sin(x^2-y^2) - 2y^2\cos(x^2-y^2)\right]$$
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = 4xy\cos(x^2-y^2) + 6x^2y - 2.$$

(d) $1 + xy^2 > 0$, pak

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^2}{\sqrt{1+xy^2}}, \quad \frac{\partial f}{\partial y}(x,y) = \frac{xy}{\sqrt{1+xy^2}},$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{-y^4}{4(1+xy^2)^{\frac{3}{2}}}, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{x}{(1+xy^2)^{\frac{3}{2}}},$$
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{y(xy^2+2)}{2(1+xy^2)^{\frac{3}{2}}}.$$