# Linear matrix inequalities and semidefinite programming and their applications in control

Graduate course on Optimal and Robust Control

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$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq b$$

Example:

$$3x_1 + 6x_2 \geq 6$$

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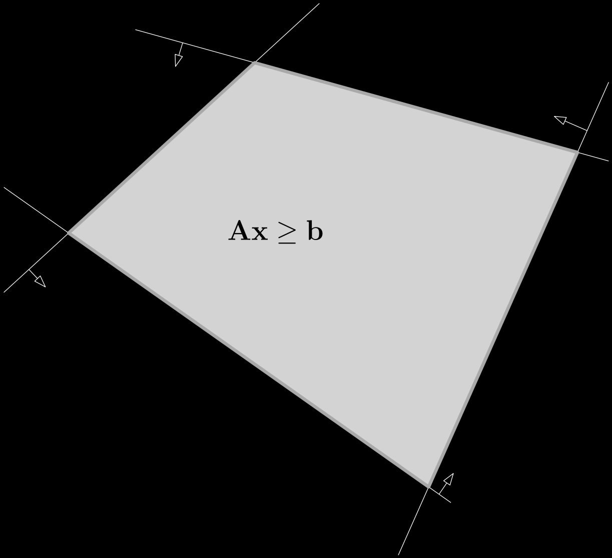
$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}} \geq \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\mathbf{b}}$$

$$\boxed{\mathbf{Ax} \geq \mathbf{b}}$$

# Solution set is convex—polyhedron/polytope



$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq b \end{aligned}$$

Primal form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = b \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Dual form

$$\begin{aligned} \text{maximize} \quad & \mathbf{b}^\mathsf{T}\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^\mathsf{T}\mathbf{y} \leq c \end{aligned}$$

# What is a linear matrix inequality (LMI)?

$$\boxed{\underbrace{\mathbf{F}_0 + \mathbf{F}_1 x_1 + \mathbf{F}_2 x_2 + \ldots + \mathbf{F}_m x_m}_{\mathbf{F}(\mathbf{x})} \succ 0,}$$

where

$$\mathbf{F}_i = \mathbf{F}_i^{\mathrm{T}} \in \mathbb{R}^{n\times n}, \quad i = 0, 1, 2, \ldots, m$$

and

$$\mathbf{F}(\mathbf{x}) \succeq 0 \qquad \text{(also } \mathbf{F}(\mathbf{x}) > 0\text{)}$$

reads that $\mathbf{F}(\mathbf{x})$ is positive definite.
Can also have the *nonstrict* inequality $F(x) \succeq 0$ (or $\geq$) for positive semidefinite.

$$\boxed{\mathbf{A} \succ 0 \quad \Leftrightarrow \quad \mathbf{x}^\top \mathbf{A} \mathbf{x} > 0,\ \forall \mathbf{x} \in \mathbb{R}^n \backslash \mathbf{0}}$$

Only symmetrix matrices

$$\begin{aligned} \mathbf{x}^\top \mathbf{A} \mathbf{x} &= \frac{1}{2}\mathbf{x}^\top \mathbf{A} \mathbf{x} + \left(\frac{1}{2}\mathbf{x}^\top \mathbf{A} \mathbf{x}\right)^\top \\ &= \mathbf{x}^\top \left(\frac{\mathbf{A} + \mathbf{A}^\top}{2}\right) \mathbf{x} \end{aligned}$$

$$\mathbf{A} \succ 0 \quad \Leftrightarrow \quad \lambda_i(\mathbf{A}) > 0,\ i = 1, \ldots, n$$

$\mathbf{A} \succ 0 \quad \Leftrightarrow \quad$ all leading principle minors positive

$$\boxed{\mathbf{A} \succ 0 \quad \Leftrightarrow \quad \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} > 0,\ \forall \mathbf{x} \in \mathbb{R}^n \backslash \mathbf{0}}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_2 \succ 0$$

$$\begin{bmatrix} x_1 + 1 & 0 & x_2 \\ 0 & 2 & -x_1 - 1 \\ x_2 & -x_1 - 1 & 2 \end{bmatrix} \succ 0$$

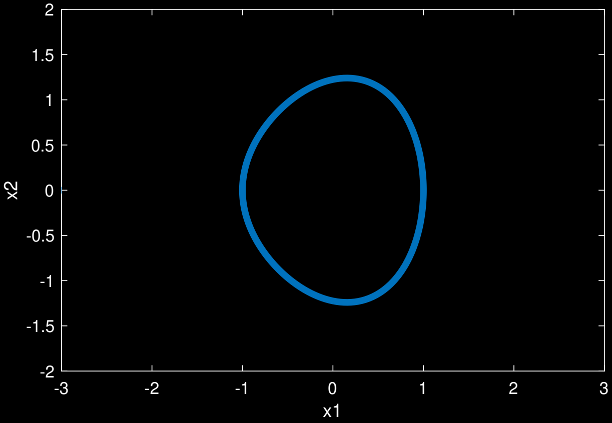
$$\begin{bmatrix}1 & 0 & 0\\0 & 2 & -1\\0 & -1 & 2\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix} x_1 + \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\1 & 0 & 0\end{bmatrix} x_2 \succ 0$$

$$\begin{bmatrix}x_1 + 1 & 0 & x_2\\0 & 2 & -x_1 - 1\\x_2 & -x_1 - 1 & 2\end{bmatrix} \succ 0$$

# Example of an LMI – solution set is convex

Leading principal minors

$$d_1 = x_1 + 1.0 > 0$$
$$d_2 = 2x_1 + 2 > 0$$
$$d_3 = -x_1^3 - 3x_1^2 - 2x_2^2 + x_1 + 3 > 0$$



# Semidefinite program (SDP)

Linear optimization with LMI constraints

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{F}_0 + \mathbf{F}_1 x_1 + \mathbf{F}_2 x_2 + \ldots + \mathbf{F}_m x_m > 0 \end{aligned}$$

In control we prefer the matrix form

$$\mathbf{F}_0 + \mathbf{G}_1\mathbf{X}_1\mathbf{H}_1 + \mathbf{G}_2\mathbf{X}_2\mathbf{H}_2 + \ldots + \mathbf{G}_k\mathbf{X}_k\mathbf{H}_k \succ 0$$

$$\mathbf{X}_i \succ 0, \quad i = 1, \ldots, k.$$

LMIs can be concatenated

$$\mathbf{F}_0 + \mathbf{G}_1\mathbf{X}_1\mathbf{H}_1 + \ldots + \mathbf{G}_k\mathbf{X}_k\mathbf{H}_k \succ 0$$
$$\mathbf{J}_0 + \mathbf{K}_1\mathbf{X}_1\mathbf{L}_1 + \ldots + \mathbf{K}_k\mathbf{X}_k\mathbf{L}_k \succ 0$$

can be written as

$$\begin{bmatrix} \mathbf{F}_0 & 0 \\ 0 & \mathbf{J}_0 \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 & 0 \\ 0 & \mathbf{K}_1 \end{bmatrix} \mathbf{X}_1 \begin{bmatrix} \mathbf{H}_1 & 0 \\ 0 & \mathbf{L}_1 \end{bmatrix} + \ldots + \begin{bmatrix} \mathbf{G}_k & 0 \\ 0 & \mathbf{K}_k \end{bmatrix} \mathbf{X}_k \begin{bmatrix} \mathbf{H}_k & 0 \\ 0 & \mathbf{L}_k \end{bmatrix} \succ 0$$

Primal form

$$\begin{aligned} \text{minimize} \quad & \langle \mathbf{C}, \mathbf{X} \rangle \\ \text{subject to} \quad & \langle \mathbf{A}_i, \mathbf{X} \rangle = b_i, \quad i = 1, \ldots, m \\ & \mathbf{X} \succ 0, \end{aligned}$$

where $\langle \mathbf{C}, \mathbf{X} \rangle = \text{Trace}(\mathbf{C}^\mathsf{T} \mathbf{X})$.

Solvers

- Sedumi `http://sedumi.ie.lehigh.edu/`
- SDPT3 `https://blog.nus.edu.sg/mattohkc/softwares/sdpt3/`
- SDPA `http://sdpa.sourceforge.net/`
- CSDP `https://github.com/coin-or/Csdp`
- ...

Interfaces

- Yalmip (Matlab) `https://yalmip.github.io/`
- CVX (Matlab) `http://cvxr.com/cvx/`
- CVXPY (Python) `https://www.cvxpy.org/`
- Convex.jl (Julia) `https://github.com/JuliaOpt/Convex.jl`
- JuMP.jl (Julia ) `https://github.com/JuliaOpt/JuMP.jl`
- ...

# Types of LMI/SDP problems

- LMI feasibility problem
- linear optimization problem
- generalized eigenvalue problem

$$\begin{aligned} \mathbf{F}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) &\succ 0 \\ \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k &\succ 0 \end{aligned}$$

Asymptotic stability of an LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

amounts to asking if there is

$$\mathbf{X} = \mathbf{X}^{\mathrm{T}} \succ 0$$

solving the Lyapunov equation

$$\mathbf{A}^{\mathrm{T}}\mathbf{X} + \mathbf{X}\mathbf{A} = -\mathbf{Y}$$

for some

$$\mathbf{Y} = \mathbf{Y}^{\mathrm{T}} \succ 0.$$

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Can be formulated as an LMI feasibility problem

$$\begin{aligned} \mathbf{A}^{\mathrm{T}}\mathbf{X} + \mathbf{X}\mathbf{A} &\prec 0 \\ \mathbf{X} &\succ 0. \end{aligned}$$

$$\begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}^{\mathrm{T}}\mathbf{X} - \mathbf{X}\mathbf{A} \end{bmatrix} \succ 0.$$

Can be formulated as an LMI feasibility problem

$$\mathbf{A}^{\mathrm{T}}\mathbf{X} + \mathbf{X}\mathbf{A} \prec 0$$
$$\mathbf{X} \succ 0.$$

$$\begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}^{\mathrm{T}}\mathbf{X} - \mathbf{X}\mathbf{A} \end{bmatrix} \succ 0.$$

```
A = [-1 2;-3 -4];
X = sdpvar(2,2);
```

```
C = [X >= I, A'*X+X*A <= -I];
```

```
solvesdp(C);
```

```
>> X_feasible = double(X)
X_feasible =
    4.4811    0.2602
    0.2602    2.2100

>> eig(X_feasible)
ans =
    2.1806
    4.5105
```

# In Matlab using Yalmip

```
A = [-1 2;-3 -4];
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```

```
A = [-1 2;-3 -4];

cvx_begin sdp
  variable X(n,n) symmetric
  A'*X + X*A <= -eye(n)
  X >= eye(n)
cvx_end
```

```
>> X
X =
    3.7179    0.3650
    0.3650    1.6490

>> eig(X)
ans =
    1.5865
    3.7804
```

Linear time-varying system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t),$$

where

$$\mathbf{A}(t) = \sum_{j=1}^{k} \alpha_j(t)\, \mathbf{A}_j$$

and

$$\sum_{j=1}^{k} \alpha_j(t) = 1, \qquad \alpha_j(t) \geq 0 \qquad \forall t \in \mathbb{R}.$$

Finding Lyapunov function for all possible (infinite number, even uncountable number of) matrices $\mathbf{A}(t) = \sum_{j=1}^{k} \alpha_j(t)\, \mathbf{A}_j$?
Checking stability of the vertices enough.
But single Lyapunov function (matrix) searched for

$$\begin{bmatrix} \mathbf{X} & & & & \\ & -\mathbf{A}_1^{\mathrm{T}}\mathbf{X} - \mathbf{X}\mathbf{A}_1 & & & \\ & & -\mathbf{A}_2^{\mathrm{T}}\mathbf{X} - \mathbf{X}\mathbf{A}_2 & & \\ & & & \ddots & \\ & & & & -\mathbf{A}_k^{\mathrm{T}}\mathbf{X} - \mathbf{X}\mathbf{A}_k \end{bmatrix} \succ 0$$

$$\begin{aligned} \underset{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k}{\text{minimize}} \quad & f(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) \\ \text{subject to} \quad & \mathbf{F}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) \succ 0 \\ & \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k \succ 0 \end{aligned}$$

## LP as an SDP

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{b} \geq \mathbf{0}. \end{aligned}$$

$$\begin{bmatrix} a_{11} & & \ldots & \\ & a_{21} & \ldots & \\ & & \ddots & \\ & & \ldots & a_{m1} \end{bmatrix} x_1 + \ldots + \begin{bmatrix} a_{1n} & & \ldots & \\ & a_{2n} & \ldots & \\ & & \ddots & \\ & & \ldots & a_{mn} \end{bmatrix} x_n + \begin{bmatrix} b_1 & & \ldots & \\ & b_2 & \ldots & \\ & & \ddots & \\ & & \ldots & b_m \end{bmatrix} \succ 0$$

For

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\mathsf{T} & \mathbf{C} \end{bmatrix}$$

Schur complement of **M** with respect to **A** is

$$\mathbf{M}/\mathbf{A} = \mathbf{C} - \mathbf{B}^\mathsf{T}\mathbf{A}^{-1}\mathbf{B}$$

and Schur complement of **M** with respect to **C** is

$$\mathbf{M}/\mathbf{C} = \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\mathsf{T}$$

$$\boxed{\mathbf{M} \succ 0 \quad \Leftrightarrow \quad \mathbf{A} \succ 0 \quad \& \quad \mathbf{C} - \mathbf{B}^\mathsf{T}\mathbf{A}^{-1}\mathbf{B} \succ 0}$$

$$\boxed{\mathbf{M} \succ 0 \quad \Leftrightarrow \quad \mathbf{C} \succ 0 \quad \& \quad \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\mathsf{T} \succ 0}$$

$$\begin{aligned} \text{minimize} \quad & (\mathbf{A}_0\mathbf{x}+\mathbf{b}_0)^{\mathrm{T}}(\mathbf{A}_0\mathbf{x}+\mathbf{b}_0) - \mathbf{c}_0^{\mathrm{T}}\mathbf{x} - d_0 \\ \text{subject to} \quad & (\mathbf{A}\mathbf{x}+\mathbf{b})^{\mathrm{T}}(\mathbf{A}\mathbf{x}+\mathbf{b}) - \mathbf{c}^{\mathrm{T}}\mathbf{x} - d \leq \mathbf{0}. \end{aligned}$$

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n,\ \gamma\in\mathbb{R}}{\text{minimize}} \quad & \gamma \\ \text{subject to} \quad & \begin{bmatrix} \mathbf{I} & \mathbf{A}_0\mathbf{x}+\mathbf{b}_0 \\ (\mathbf{A}_0\mathbf{x}+\mathbf{b}_0)^{\mathrm{T}} & \mathbf{c}_0^{\mathrm{T}}\mathbf{x}+d_0+\gamma \end{bmatrix} \succ 0 \\ & \begin{bmatrix} \mathbf{I} & \mathbf{A}\mathbf{x}+\mathbf{b} \\ (\mathbf{A}\mathbf{x}+\mathbf{b})^{\mathrm{T}} & \mathbf{c}^{\mathrm{T}}\mathbf{x}+d \end{bmatrix} \succ 0 \end{aligned}$$

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \lambda_{\max}(\mathbf{A}(\mathbf{x})) \\ \text{subject to} \quad & \mathbf{B}(\mathbf{x}) \succ 0 \end{aligned}$$

$$\begin{aligned} \underset{\lambda\in\mathbb{R},\, \mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & \lambda\mathbf{I} - \mathbf{A}(\mathbf{x}) \succ 0 \\ & \mathbf{B}(\mathbf{x}) \succ 0. \end{aligned}$$

## Minimizing the maximum eigenvalue

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \lambda_{\max}(\mathbf{A}(\mathbf{x})) \\ \text{subject to} \quad & \mathbf{B}(\mathbf{x}) \succ 0 \end{aligned}$$

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$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad \|\mathbf{A}(\mathbf{x})\|_2$$

$$\begin{aligned} \underset{\gamma\in\mathbb{R},\ \mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \gamma \\ \text{subject to} \quad & \begin{bmatrix} \gamma\mathbf{I} & \mathbf{A}(\mathbf{x}) \\ \mathbf{A}^{\mathrm{T}}(\mathbf{x}) & \mathbf{I} \end{bmatrix} \succeq 0. \end{aligned}$$

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Solving

$$\mathbf{Ax} \approx \mathbf{b}$$

so that the $\|.\|_\infty$ is minimized.
That is, solve the optimization problem

$$\min_{\mathbf{x}} \max_{i} \left|\mathbf{A}_{(i,:)}\mathbf{x} - b_i\right|$$

In some application it is more suitable to work in logarithmic scales

$$\min_{x} \max_{i} \left|\log(\mathbf{A}_{(i,:)}\mathbf{x}) - \log(b_i)\right|$$

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$$\text{minimize}\, \lambda$$

subject to

$$\begin{aligned} \mathbf{F}_1(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k) - \lambda \mathbf{F}_2(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k) &\prec 0 \\ \mathbf{F}_2(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k) &\succ 0 \\ \mathbf{F}_3(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k) &\succ 0 \end{aligned}$$

Is quasiconvex.

# Generalized eigenvalue problem (GEVP)

$$\text{minimize}\, \lambda$$

subject to

$$\begin{aligned} \mathbf{F}_1(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) - \lambda \mathbf{F}_2(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) &\prec 0 \\ \mathbf{F}_2(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) &\succ 0 \\ \mathbf{F}_3(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) &\succ 0 \end{aligned}$$

Is quasiconvex.

# Ex.: upper bound on $\mu$ (structured singular value, SSV).

For a given matrix $\mathbf{M}$, find a diagonal matrix $\mathbf{D}$ such that $\|\mathbf{D}\mathbf{M}\mathbf{D}^{-1}\|_2$ is minimized.

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For a given matrix $\mathbf{M}$, find a diagonal matrix $\mathbf{D}$ such that $\|\mathbf{D}\mathbf{M}\mathbf{D}^{-1}\|_2$ is minimized.

Does there exist a solution $x$ to the quadratic inequality

$$F_0(\mathbf{x}) := \mathbf{x}^{\mathrm{T}}\mathbf{A}_0\mathbf{x} + 2\mathbf{b}_0^{\mathrm{T}}\mathbf{x} + c_0 \geq 0$$

for all **x** satisfying another quadratic inequality

$$F_1(\mathbf{x}) := \mathbf{x}^{\mathrm{T}}\mathbf{A}_1\mathbf{x} + 2\mathbf{b}_1^{\mathrm{T}}\mathbf{x} + c_1 \geq 0?$$

We can consider just one inequality

$$F_1(\mathbf{x}) - \tau F_2(\mathbf{x}) \geq 0$$

for some $\tau \geq 0$.
(Showing sufficiency easy, necessity for two inequalities difficult.
For more inequalities somewhat conservative.)

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Consider the inequality in two variables $\mathbf{X}$ and $\mathbf{\Lambda}$

$$\mathbf{\Psi}(\mathbf{X}) + \mathbf{G}(\mathbf{X})\mathbf{\Lambda}\mathbf{H}^{\mathrm{T}}(\mathbf{X}) + \mathbf{H}(\mathbf{X})\mathbf{\Lambda}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}(\mathbf{X}) \prec 0$$

and note that $\mathbf{G}$ and $\mathbf{H}$ are not functions of $\mathbf{\Lambda}$.
Denote $\mathbf{N}_{\mathbf{G}}$ and $\mathbf{N}_{\mathbf{H}}$ matrices whose columns are bases of nullspace of $\mathbf{G}$ and $\mathbf{H}$, respectively.
Then the original LMI is solvable iff

$$\mathbf{N}_{\mathbf{G}}^{\mathrm{T}}\mathbf{\Psi}(\mathbf{X})\mathbf{N}_{\mathbf{G}} \prec 0$$
$$\mathbf{N}_{\mathbf{H}}^{\mathrm{T}}\mathbf{\Psi}(\mathbf{X})\mathbf{N}_{\mathbf{H}} \prec 0$$

# Projection lemma (also Elimination lemma

Consider the inequality in two variables $\mathbf{X}$ and $\boldsymbol{\Lambda}$

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$$\mathbf{\Psi}(\mathbf{X}) + \mathbf{G}(\mathbf{X})\mathbf{\Lambda}\mathbf{H}^{\mathrm{T}}(\mathbf{X}) + \mathbf{H}(\mathbf{X})\mathbf{\Lambda}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}(\mathbf{X}) \prec 0$$

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$$\mathbf{N_H}^{\mathrm{T}}\mathbf{\Psi}(\mathbf{X})\mathbf{N_H} \prec 0$$

# Ex.: application of projection lemma to state feedback stabilization

Find (if possible) $\mathbf{X} \succ 0$ such that

$$(\mathbf{A} + \mathbf{BK})^{\mathrm{T}}\mathbf{X} + \mathbf{X}(\mathbf{A} + \mathbf{BK}) \prec 0.$$

Using substitution, modify as

$$\mathbf{YA}^{\mathrm{T}} + \mathbf{AY} + \mathbf{YK}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} + \mathbf{BKY} \prec 0$$

It is equivalent to

$$\mathbf{N}_{\mathbf{B}}^{T}(\mathbf{YA}^{\mathrm{T}} + \mathbf{AY})\mathbf{N}_{\mathbf{B}} \prec 0$$
$$\mathbf{N}_{\mathbf{I}}^{T}(\mathbf{YA}^{\mathrm{T}} + \mathbf{AY})\mathbf{N}_{\mathbf{I}} \prec 0$$

However, $\mathbf{N}_{\mathbf{I}} = 0$, hence the second equation can be omitted.

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Using substitution, modify as

$$\mathbf{Y}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} + \mathbf{B}\mathbf{K}\mathbf{Y} \prec 0$$

It is equivalent to

$$\mathbf{N}_{\mathbf{B}}^{T}(\mathbf{Y}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{Y})\mathbf{N}_{\mathbf{B}} \prec 0$$
$$\mathbf{N}_{\mathbf{I}}^{T}(\mathbf{Y}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{Y})\mathbf{N}_{\mathbf{I}} \prec 0$$

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