① a) $i \in M \Rightarrow f(M) = L \cup \{\infty\}$, kde $L \subseteq \mathbb{C}$ je přímka.

$$\begin{tikzcd} \infty \arrow[r, "f"] \arrow[d, leftrightarrow, "M"'] & 2 \arrow[d, leftrightarrow, "f(M)"] \\ 0 \arrow[r, "f"] & -1 \end{tikzcd}$$

$\Rightarrow L$ je osa úsečky $seg(-1;2)$

tj. $L = \{z \in \mathbb{C} \mid \operatorname{Re} z = \frac{1}{2}\}$.

b) Ať $K = \partial N$. Protože $i \notin K$, je $f(K)$ kružnice se středem S a poloměrem R.

$$\begin{tikzcd} i \arrow[r, "f"] \arrow[d, leftrightarrow, "K"'] & \infty \arrow[d, leftrightarrow, "f(K)"] \\ \frac{4}{-i} \arrow[r, "f"] & S \end{tikzcd}$$

$\Rightarrow S = f(4i) = \frac{9i}{3i} = 3$

$R = |f(2i) - S| = 2$.

Protože $f(i) = \infty$ a $i \in \operatorname{Int} K$, je

$$f(N) = \{z \in \mathbb{C} \mid |z-3| \geq 2\} \cup \{\infty\}.$$

② a) $f(z) = 1 - \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} z^{4n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} z^{4n}$

pro $z \in \mathbb{C}$ (tj. $R = +\infty$).

b) $g(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} z^{4n-k}$ pro $z \in \mathbb{C} \setminus \{0\}$

- pro $k \in \{1,2,3,4\}$ má $g(z)$ v bodě 0 odstranitelnou singularitu.
- Ať $k \geq 5$. Protože $\left. \frac{(-1)^{n+1} 4^n}{(2n)!} \right|_{n=1} \neq 0$, má $g(z)$ v 0 pól řádu $k-4$.

Z výše nalezeného rozvoje funkce $g(z)$ vidíme, že

- je-li $k=8$, pak $\operatorname{res}_0 g(z) = 0$;
- je-li $k=3$, pak

$$\operatorname{res}_0 g(z) = \left. \frac{(-1)^{n+1} 4^n}{(2n)!} \right|_{n=2} = \frac{-16}{4!} = -\frac{2}{3}.$$

③ a) $A(z) = \mathcal{Z}[i^n](z)\,\mathcal{Z}[\cos(\frac{n\pi}{2})](z)$

$$= \frac{z}{z-i}\,\frac{z^2}{z^2+1} = \frac{z^3}{(z-i)^2(z+i)}$$

b) $\operatorname{res}_i A(z)z^{n-1} = \lim_{z\to i}\left(\frac{z^{n+2}}{z+i}\right)'$

$$= \lim_{z\to i}\frac{(n+2)z^{n+1}(z+i) - z^{n+2}}{(z+i)^2} = \frac{2(n+2)i^{n+2} - i^{n+2}}{-4}$$

$$= \frac{2n+3}{4}\,i^n$$

$$\operatorname{res}_{-i} A(z)z^{n-1} = \frac{(-i)^{n+2}}{-4} = \frac{(-i)^n}{4}$$

$$\Rightarrow a_n = \frac{(-i)^n}{4} + \frac{2n+3}{4}\,i^n \quad \text{pro } n\in\mathbb{N}_0.$$

c) $\mathcal{Z}[a_{n+2}](z) = z^2\left[A(z) - a_0 - \frac{a_1}{z}\right]$

$$= z^2A(z) - z^2 - iz.$$

(4) $$-(\omega^2+1)\hat{g}(\omega) = -\frac{1}{\omega^2+16}$$

$$\hat{g}(\omega) = \frac{1}{(\omega^2+1)(\omega^2+16)}$$

Ať $f(z) = \dfrac{e^{izt}}{(z^2+1)(z^2+16)}$.

$$\operatorname{res}_{\pm i} f(z) = \frac{e^{izt}}{2z(z^2+16)+2z(z^2+1)}\bigg|_{z=\pm i} = \frac{e^{\mp t}}{\pm 30 i}$$

$$\operatorname{res}_{\pm 4i} f(z) = \quad -\,\|\,- \quad \bigg|_{z=\pm 4i} = \frac{e^{\mp 4t}}{\mp 120 i}$$

Pro $t \geq 0$ je

$$g(t) = \frac{1}{2\pi} 2\pi i \left(\operatorname{res}_{i} f(z) + \operatorname{res}_{4i} f(z)\right) = \frac{4e^{-t} - e^{-4t}}{120}.$$

Pro $t < 0$ je

$$g(t) = \frac{1}{2\pi}(-2\pi i)\left(\operatorname{res}_{-i} f(z) + \operatorname{res}_{-4i} f(z)\right) = \frac{4e^{t} - e^{4t}}{120}$$

Tedy

$$g(t) = \frac{1}{30} e^{-|t|} - \frac{1}{120} e^{-4|t|} \quad \text{pro } t \in \mathbb{R}.$$