① a) $$\left.\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{-2xy}{(1+x^2-y)^2} \\ \frac{\partial f}{\partial y}(x,y) &= \frac{1+x^2\cancel{-y+y}}{(1+x^2-y)^2}\end{aligned}\right\} \Rightarrow \nabla f(1,1) = (-2,2)$$

Tečná rovina má proto rovnici

$$z = 1 - 2(x-1) + 2(y-1).$$

b) Normálové vektory uvedených dvou rovin jsou: $n_1 = (-2, 2, -1)$

$$n_2 = (1, 0, 1)$$

$$\Rightarrow \cos\varphi = \frac{|n_1 \cdot n_2|}{\|n_1\|\,\|n_2\|} = \frac{3}{3\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ hledaný úhel je

$$\varphi = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

(2) Hledáme body maxima funkce $f(x,y,z)=z$ na C.

Ať $g_1(x,y,z)=(x-1)^2+(y-1)^2+\frac{z^2}{2}-1$, $g_2(x,y,z)=x+y-z-2$.

Metoda Lagrangeových multiplikátorů vede na:

$$\left.\begin{aligned} 0+2\lambda(x-1)+\mu&=0 \quad (1)\\ 0+2\lambda(y-1)+\mu&=0 \quad (2)\end{aligned}\right\} \Rightarrow \lambda=0 \text{ nebo } x=y$$

$$1+\lambda z \qquad -\mu=0 \quad (3)$$

$$g_1(x,y,z)=0 \quad (4)$$

$$g_2(x,y,z)=0 \quad (5)$$

- $\lambda=0 \overset{(1)}{\Rightarrow} \mu=0$, což je spor s (3).
- $x=y \overset{(5)}{\Rightarrow} z=2(x-1) \overset{(4)}{\Rightarrow} 4(x-1)^2=1 \Rightarrow |x-1|=\frac{1}{2}$.

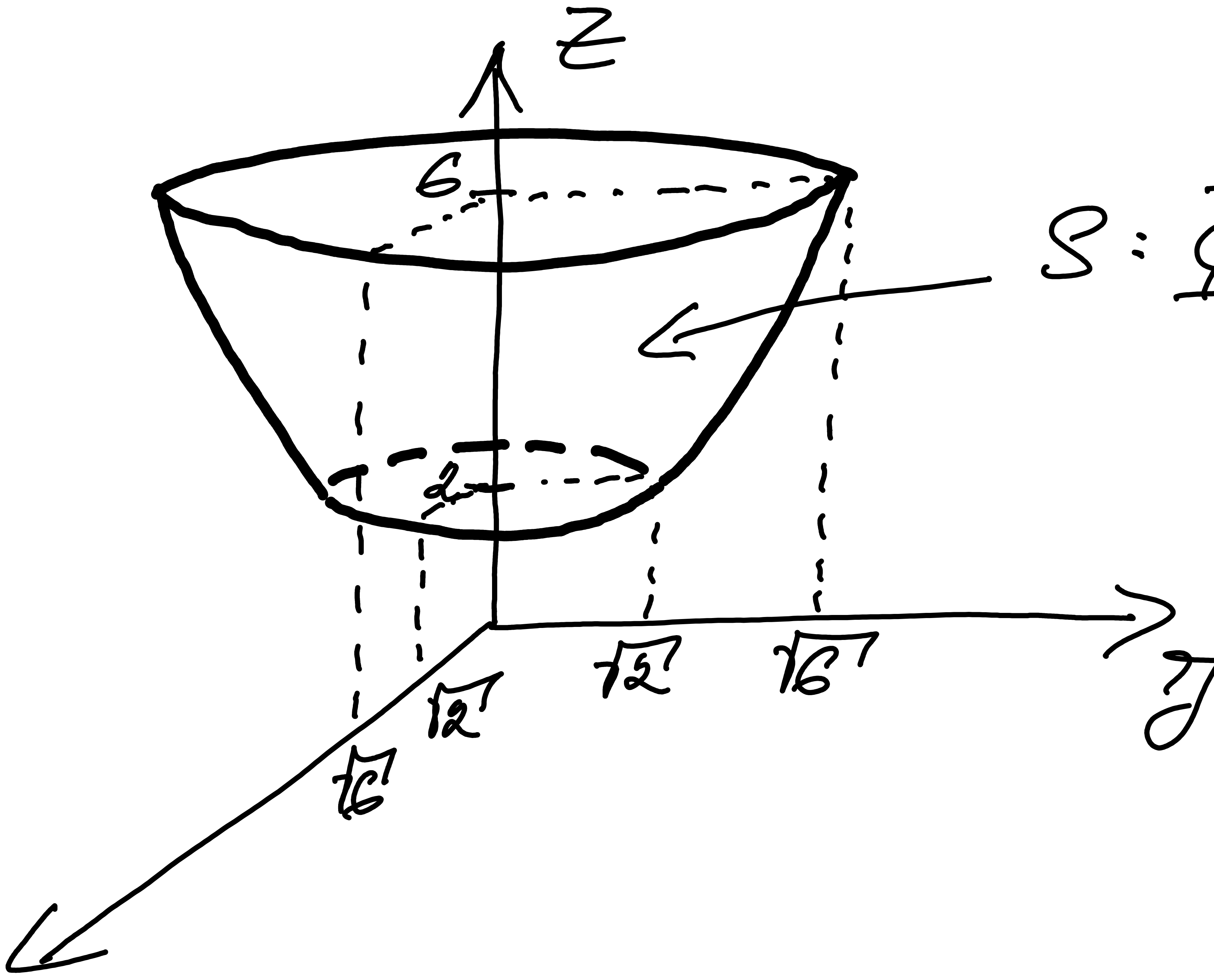
$$\Rightarrow \begin{cases} x=\frac{3}{2}=y, \; z=2\left(\frac{3}{2}-1\right)=1 \\ x=\frac{1}{2}=y, \; z=2\left(\frac{1}{2}-1\right)=\frac{1}{2} \end{cases}$$

Hledaný bod je tedy $\left(\frac{3}{2},\frac{3}{2},1\right)$.

[Pozn.: $\nabla g_1(x,y,z)=\alpha\nabla g_2(x,y,z) \Leftrightarrow$
$$\left.\begin{aligned} 2(x-1)&=\alpha\\ 2(y-1)&=\alpha\\ z&=-\alpha\end{aligned}\right\} \overset{(5)}{\Rightarrow} \alpha=0$$

Tedy $x=y=1$, $z=0$. Ale $g_1(1,1,0)\neq 0$. Proto $\nabla g_1(x,y,z)$ a $\nabla g_2(x,y,z)$ tvoří lin. nezáv. množ. $\forall (x,y,z)\in C$.]

3



$$S : \Phi(u,v) = (u\cos v, u\sin v, u^2),$$
$$u \in [\sqrt{2}, \sqrt{6}],\ v \in [0, 2\pi]$$

$$\frac{\partial \Phi}{\partial u}(u,v) \perp \frac{\partial \Phi}{\partial v}(u,v)$$

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial u}(u,v) &= (\cos v, \sin v, 2u) \\ \frac{\partial \Phi}{\partial v}(u,v) &= (-u\sin v, u\cos v, 0) \end{aligned}\right\} \Longrightarrow$$

$$\Longrightarrow \left\| \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) \right\| = \left\| \frac{\partial \Phi}{\partial u}(u,v) \right\| \left\| \frac{\partial \Phi}{\partial v}(u,v) \right\|$$

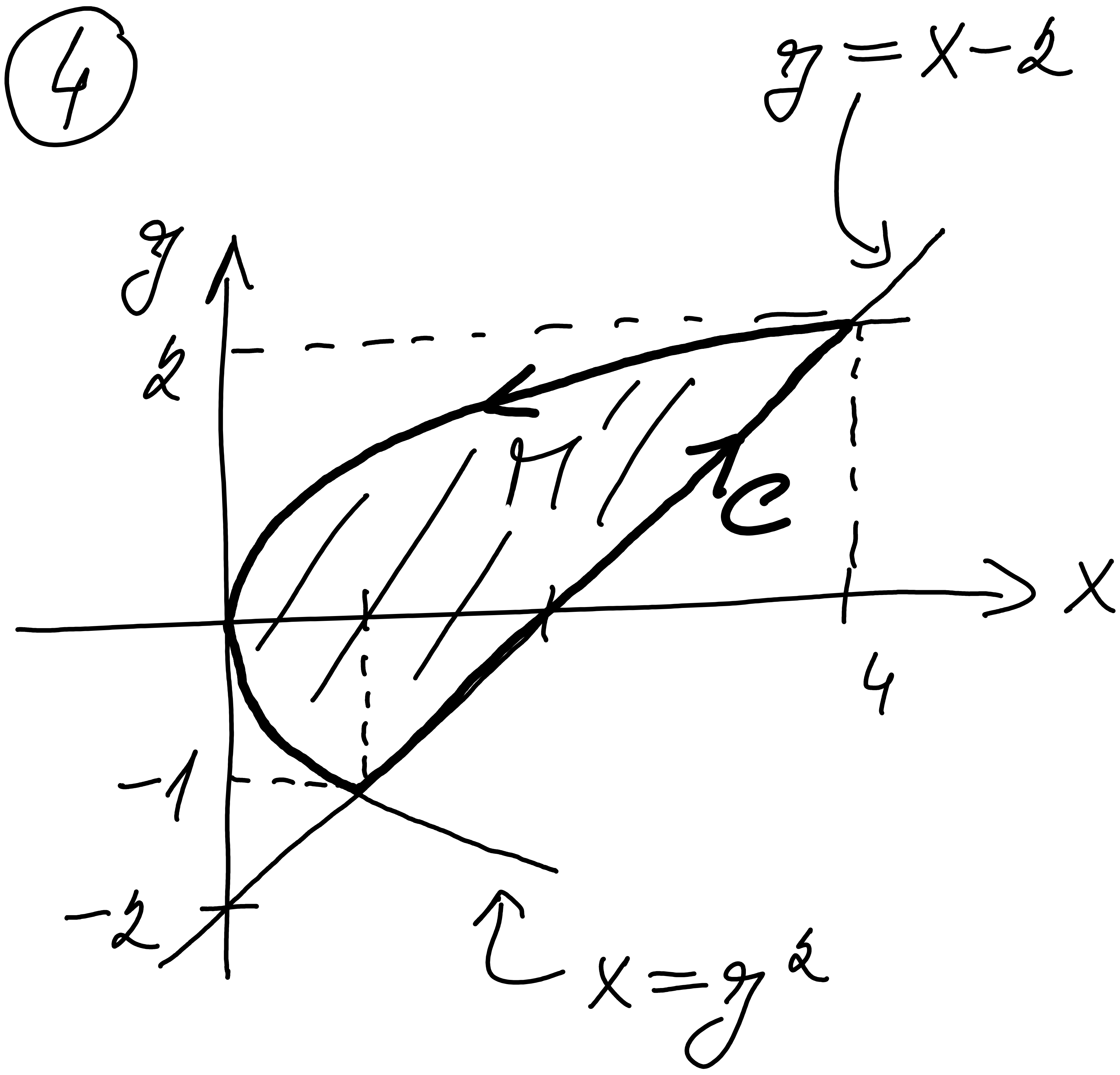
$$= \sqrt{1+4u^2} \cdot u$$

$$\Longrightarrow \text{obsah}(S) = \int_S 1\, d\sigma = \int_{\sqrt{2}}^{\sqrt{6}} \int_0^{2\pi} u\sqrt{1+4u^2}\, dv\, du$$

$$= 2\pi \int_{\sqrt{2}}^{\sqrt{6}} u\sqrt{1+4u^2}\, du = 2\pi \left[\frac{(1+4u^2)^{3/2}}{\frac{3}{2}\cdot 8} \right]_{\sqrt{2}}^{\sqrt{6}}$$

$$= 2\pi\, \frac{125-27}{12} = 2\pi\, \frac{98}{12} = \frac{49}{3}\pi.$$

4



$y = x - 2$

$\underline{x = y^2}$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

Průnik přímky $y = x - 2$ a paraboly $x = y^2$ je $\{(1,-1),(4,2)\}$

$$\int_C F(x,y)\cdot ds = \int_M \frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\, d\lambda_2(x,y)$$

$$= \int_{-1}^{2}\int_{y^2}^{y+2} y - 2xy \, dx\, dy = \int_{-1}^{2} y\left[x - x^2\right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^{2} y\left[\overbrace{y+2-(y+2)^2 - y^2 + y^4}^{y+2-y^2-4y-4-y^2+y^4}\right] dy$$

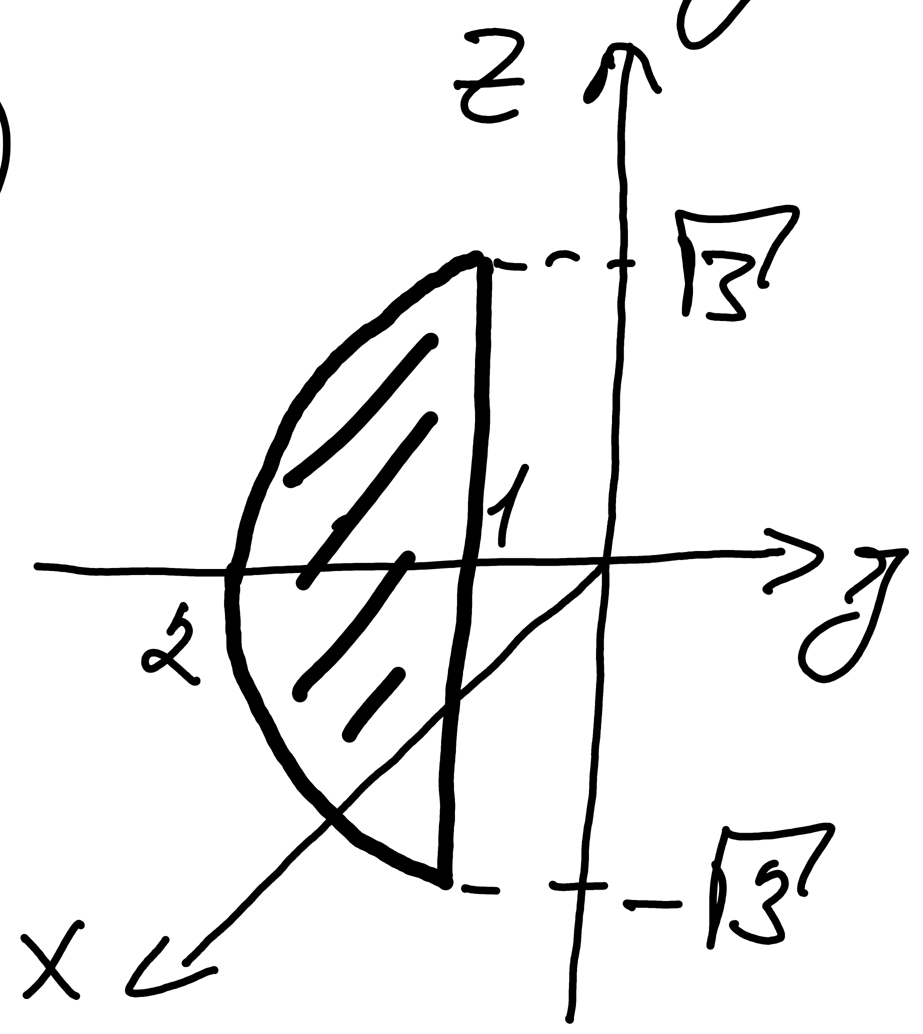
$$= \int_{-1}^{2} y^5 - 2y^3 - 3y^2 - 2y \, dy = \left[\frac{y^6}{6} - \frac{y^4}{2} - y^3 - y^2\right]_{-1}^{2}$$

$$= \frac{2^6}{6} - 16 - 4 - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{64 - 120 - 1 + 3}{6} = 11 - 20 = -9.$$

(5) $$H_f(x,y) = \begin{pmatrix} 4e^{2x-y} & -2e^{2x-y} \\ -2e^{2x-y} & e^{2x-y} \end{pmatrix}$$

Protože $4e^{2x-y} > 0$, $\det H_f(x,y) = 4e^{4x-2y} - 4e^{4x-2y} = 0$ a $e^{2x-y} > 0$, je $H_f(x,y)$ pozitivně semidef. na $\mathbb{R}^2$. Tedy f je konvexní.

(6)



- $x = 0,\ y \leq -1 \Rightarrow \varphi = \frac{3\pi}{2}$
- $\theta \in \left[\arcsin\frac{1}{2}, \pi - \arcsin\frac{1}{2}\right]$, tj. $\theta \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$
- z $\varphi = \frac{3\pi}{2}$ a $y \leq -1$ máme

$$-r\sin\theta \leq -1 \Rightarrow \frac{1}{\sin\theta} \leq r \Rightarrow r \in \left[\frac{1}{\sin\theta}, 2\right]$$

Tedy $N = \left\{(r,\varphi,\theta) \in \mathbb{R}^3 \mid r \in \left[\frac{1}{\sin\theta}, 2\right], \varphi = \frac{3\pi}{2}, \theta \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]\right\}$.

(7) $$a_k = \frac{2}{3}\int_{-2}^{1} t^3 \cos\left(\frac{2\pi k t}{3}\right) dt, \quad k \in \mathbb{N}_0,$$

$$b_k = \frac{2}{3}\int_{-2}^{1} t^3 \sin\left(\frac{2\pi k t}{3}\right) dt, \quad k \in \mathbb{N},$$

$$\mathcal{F}_f(4) = \mathcal{F}_f(1) = \frac{-8+1}{2} = -\frac{7}{2}.$$