3 – Poles, zeros, responses

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Automatic Control

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Poles

Transfer function poles

$$g(s) = b(s)/a(s)$$

$$s_i : a(s_i) = 0$$
$$g(s_i) = \infty$$

(mind zero-pole cancellation)

• subset of system poles

System poles

$$c(s) = \det\left(s\mathbf{I} - \mathbf{A}\right)$$

• eigenvalues $\lambda_i\left(\mathbf{A}\right)$
• internal dynamics, system's resonances ...
• independent on B and C matrices (i.e. sensors and actuators). Compare to transfer function poles.

Nuly

Transfer function zeros

$$G(s) = b(s)/a(s)$$

$$s_i : b(s_i) = 0$$
$$g(s_i) = 0$$

(mind zero-pole cancellations)

- blocking zeros
- always input-output related
- non-minimum phase
- complications towards controls design (unstable poles/zeros)
- collocated / noncollocated control

System zeros

$$q(s)/p(s) \qquad\qquad p(s)=\det(s\mathbf{I}-\mathbf{A})$$

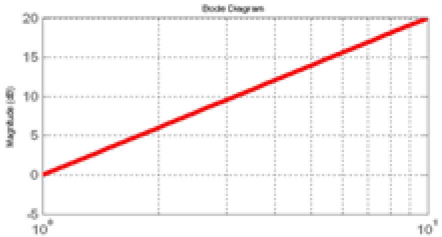
$$q(s)=\mathbf{C}\,\mathrm{adj}(s\mathbf{I}-\mathbf{A})\mathbf{B}+\det(s\mathbf{I}-\mathbf{A})\mathbf{D}=$$
$$=\det(s\mathbf{I}-\mathbf{A})\left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]=\det\begin{bmatrix} s\mathbf{I}-\mathbf{A} & \mathbf{B} \\ -\mathbf{C} & \mathbf{D}\end{bmatrix}$$

- on top of the transfer zeros (no zeros-poles cancelled):
  - input zeros (=uncontrollable poles), $z_i:\mathrm{rank}\begin{bmatrix} z_i\mathbf{I}-\mathbf{A} & \mathbf{B}\end{bmatrix}<n$
  - output zeros (unobservable poles),

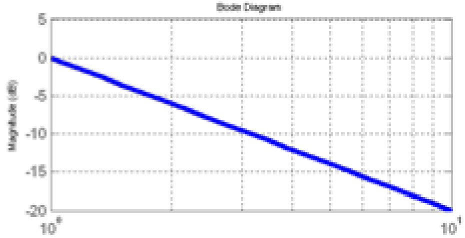
$$z_i:\mathrm{rank}\begin{bmatrix} z_i\mathbf{I}-\mathbf{A} \\ \mathbf{C}\end{bmatrix}<n$$

Poles and zeros at infinity

$$G(s) = s = \frac{s}{1} \qquad \lim_{s \to \infty} G(s) = \infty$$



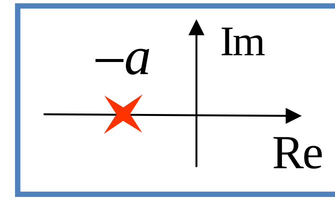
$$G(s) = \frac{1}{s} \qquad \lim_{s \to \infty} G(s) = 0$$



- realizable systems / causal systems
- relative order
- # poles = #zeros (including those at infinity)

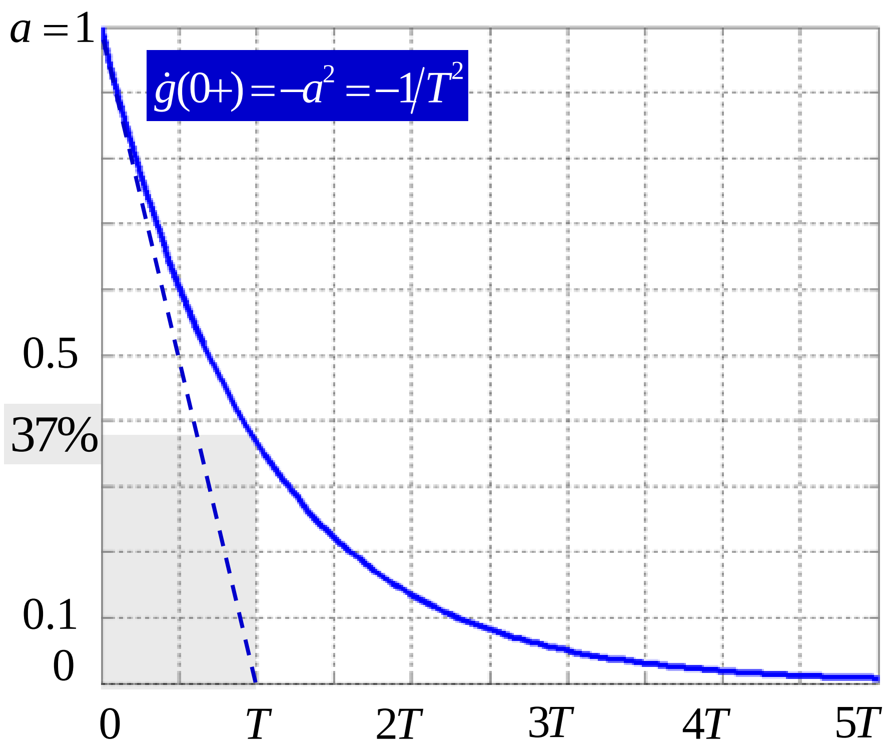
1st order system

$$G(s)=\frac{a}{s+a}=\frac{1}{1+Ts}$$



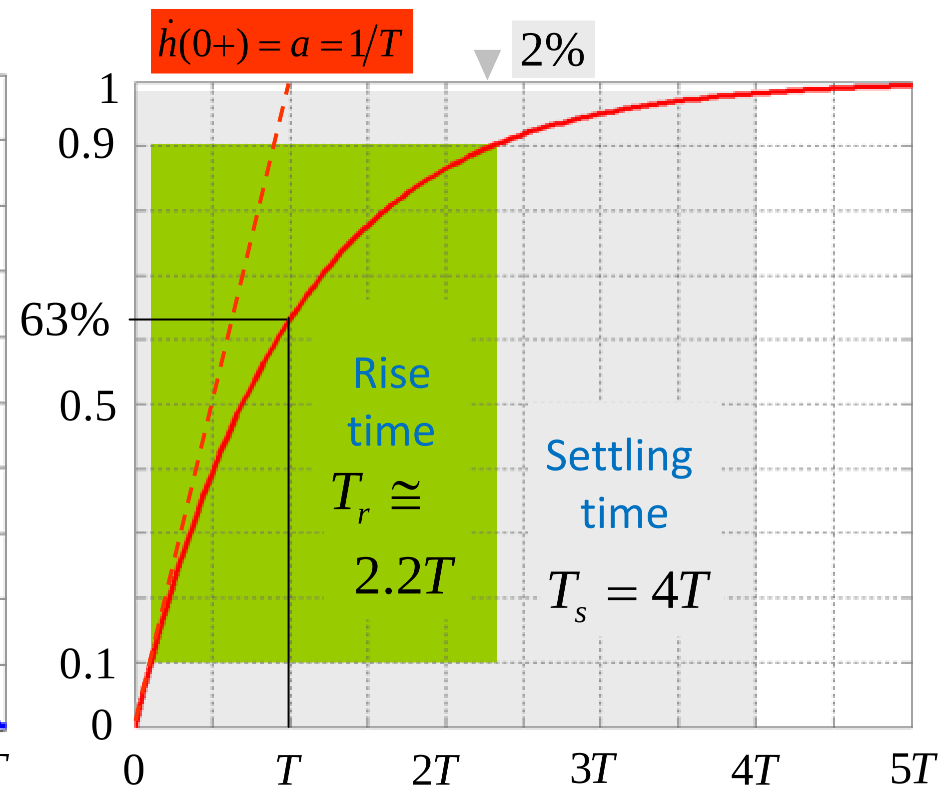
Impulse response

$$g(t)=ae^{-at}=\frac{1}{T}e^{-\frac{t}{T}}$$



Step response

$$h(t)=1-e^{-at}=1-e^{-\frac{t}{T}}$$



2^{nd} order system

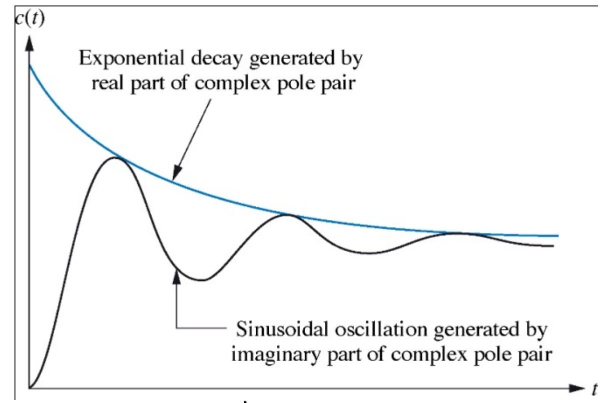
$$G(s)=\frac{b}{s^2+as+b}\quad a\ge 0,\ b>0$$

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s+\sigma-j\omega_d)(s+\sigma+j\omega_d)}=\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}=\frac{\omega_n^2}{(s+\sigma)^2+\omega_d^2}$$

(natural frequency) $\omega_n=\sqrt{b}$

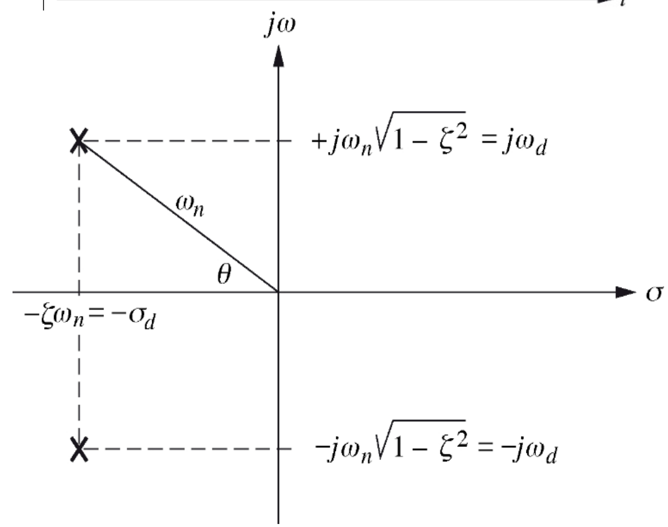
(exponential decay frequency) $\sigma=a/2$

(damping ratio) $\zeta=\dfrac{\sigma}{\omega_n}=\dfrac{1}{2\pi}\dfrac{T_n}{T_\sigma}=\dfrac{a/2}{\omega_n}=\cos\theta$



(damped frequency)

$$\omega_d=\omega_n\sqrt{1-\zeta^2}=\sqrt{b}\sqrt{1-(a/(2\sqrt{b}))^2}$$



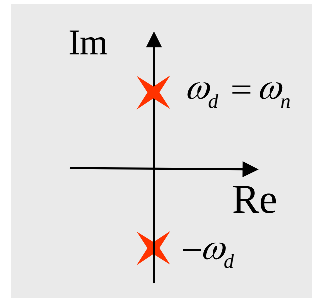
2nd order system

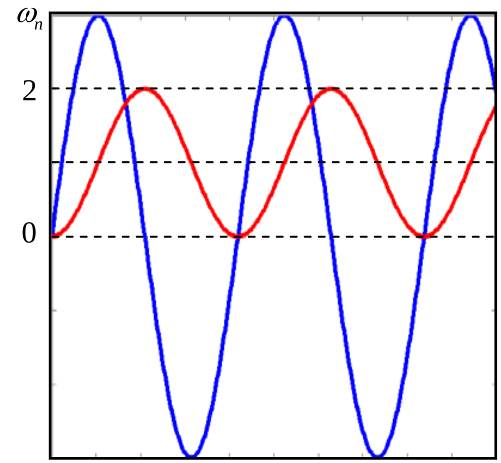
Undamped system $\sigma = 0, \omega_n = \omega_d, \zeta = 0$

$$G(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$g(t) = \omega_n \sin \omega_n t$$

$$h(t) = 1 - \cos \omega_n t$$



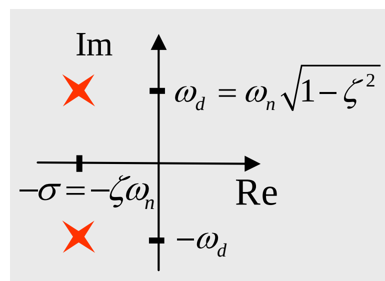


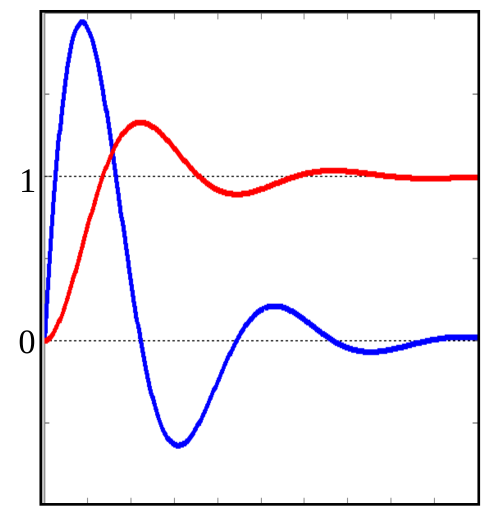
Underdamped system $|\zeta| < 1, \sigma = \zeta\omega_n, \omega_d = \omega_n\sqrt{1-\zeta^2}$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma-j\omega_d)(s+\sigma+j\omega_d)}$$

$$g(t) = (\omega_n^2/\omega_d)e^{-\sigma t}\sin \omega_d t$$

$$h(t) = 1 - e^{-\sigma t}\left[\cos \omega_d t + (\sigma/\omega_d)\sin \omega_d t\right]$$





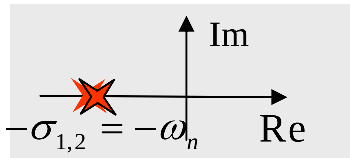
2^{nd} order system

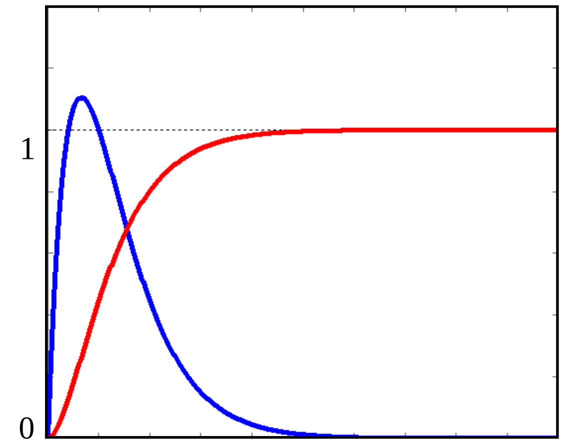
Critical damped system $\zeta = 1, \sigma_{1,2} = \omega_n, \omega_d = 0$

$$G(s) = \frac{\omega_n^2}{(s+\omega_n)^2}$$

$$g(t) = \omega_n^2 t e^{-\omega_n t}$$

$$h(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$



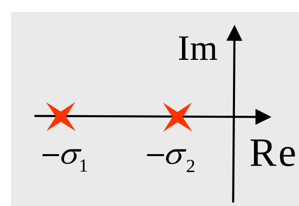


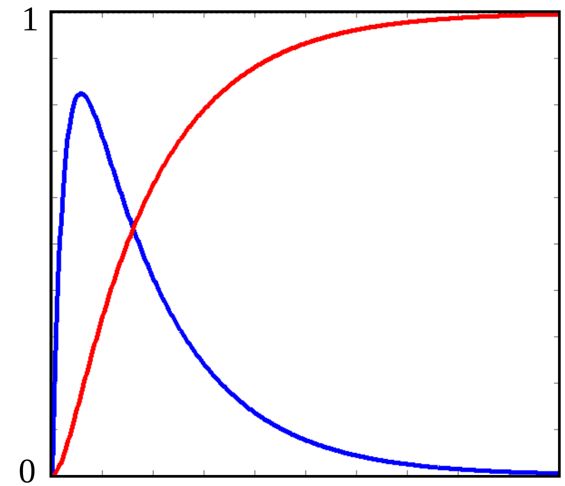
Overdamped system $|\zeta| \geq 1$:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma_1)(s+\sigma_2)}$$

$$\sigma_1 = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$\sigma_2 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

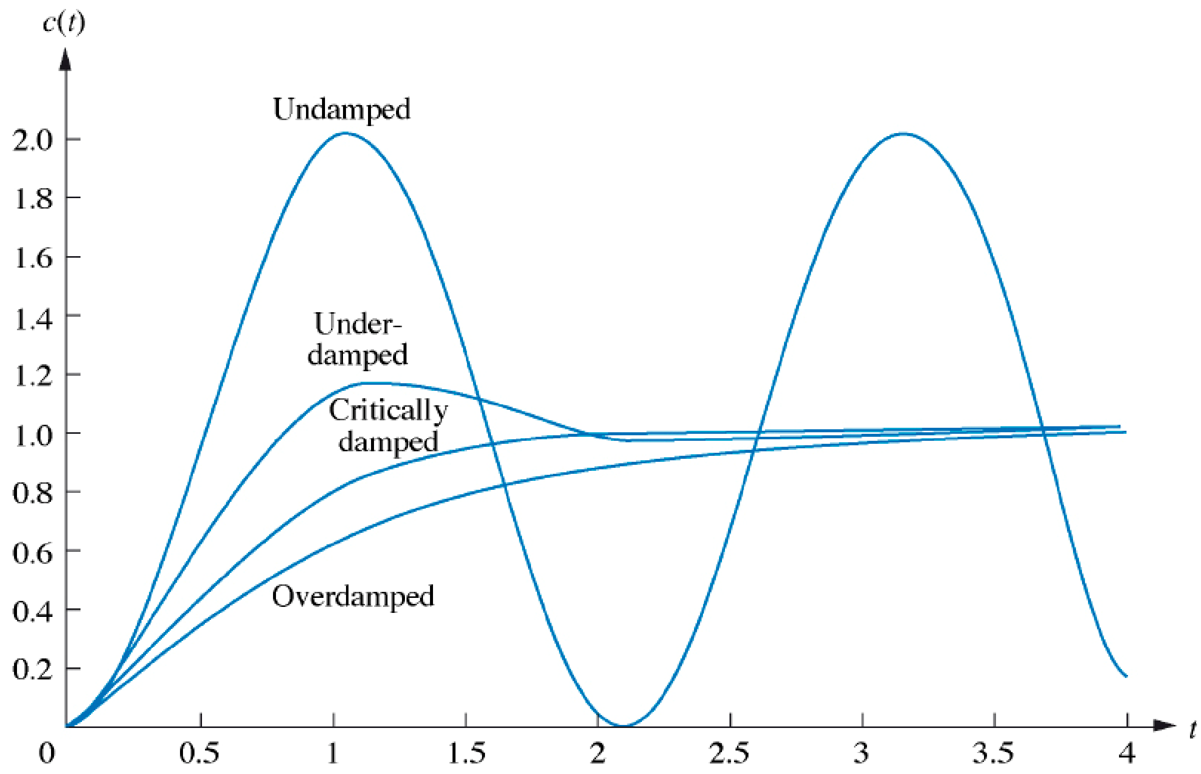


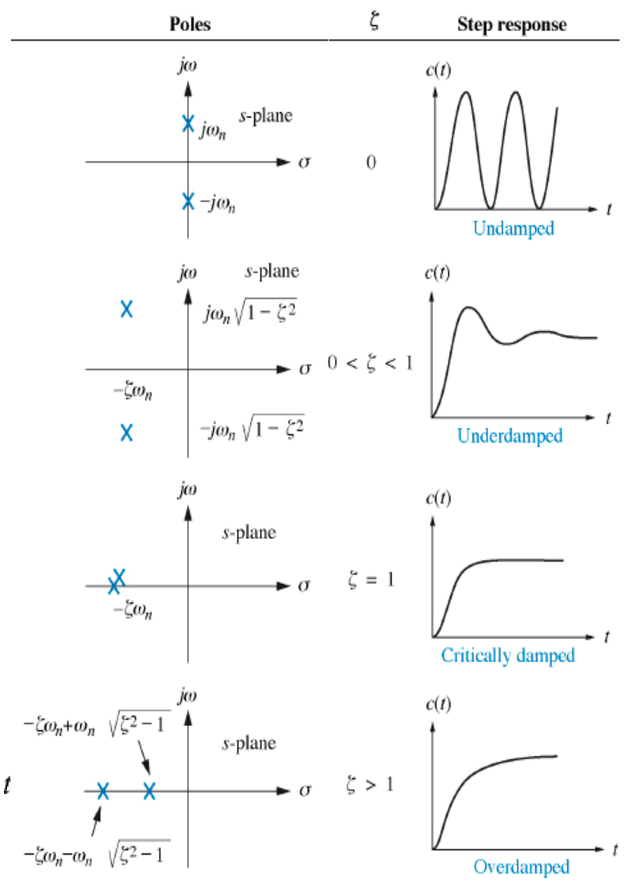


$$g(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(e^{-\sigma_2 t} - e^{-\sigma_1 t}\right)$$

$$h(t) = \frac{\sigma_1 - \sigma_2 + \sigma_2 e^{-\sigma_1 t} - \sigma_1 e^{-\sigma_2 t}}{\sigma_1 - \sigma_2}$$

2nd order system





Underdamped 2nd order system

(settling time)

$$T_s \doteq \frac{4}{\zeta\omega_n}$$

(peek time)

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

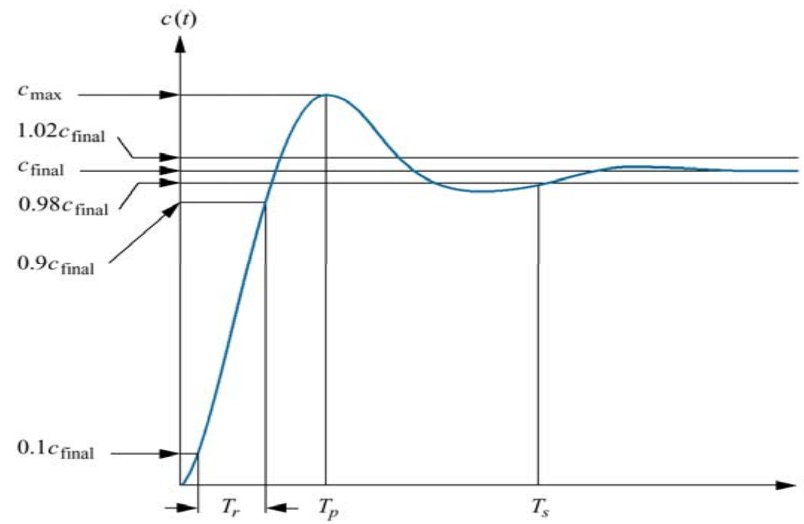
(overshoot)

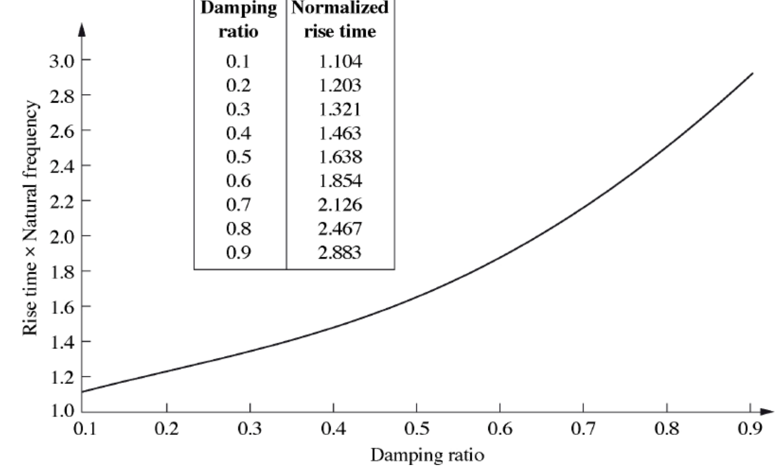
$$\%OS = 100e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)}$$

$$\zeta = \frac{-\ln\left(\%OS/100\right)}{\sqrt{\pi^2 + \ln^2\left(\%OS/100\right)}}$$

Rise time: T_r

$$T_r \approx \frac{1.8}{\omega_n}$$





Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Higher order systems. Dominant poles.

- 2nd order approximation
- dominant poles (least damped)

$$y(s)=\frac{bc}{s\left(s^2+as+b\right)\left(s+c\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+as+b}+\frac{D}{s+c}$$

$$D=\frac{-b}{c^2+b-ca},C=\frac{ca^2-c^2a-bc}{c^2+b-ca}$$

$$A=1,B=\frac{ca-c^2}{c^2+b-ca}$$

$$c\rightarrow\infty: A=1,B=-1,C=-a,D=0$$

- higher-order poles, more than 5x (10x) to the left, ignored