Exercises for lectures
5 - Identification

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2\textsuperscript{nd} order oscillatory response without zeros

- Search for
  \[ G(s) = k \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]
- Apply
  \[ u(s) = \frac{u(\infty)}{s} \]
1. Measure \( y(\infty), A_1, A_2, T_d \)
2. Calculate
  \[ k = \frac{y(\infty)}{u(\infty)}, \mu = \ln \frac{A_1}{A_2}, \zeta = \frac{\mu}{\sqrt{4\pi^2 + \mu^2}}, \omega_n = \frac{2\pi}{T_d \sqrt{1 - \zeta^2}} \]

- Terminology: \( A_1/A_2 \) attenuation factor, \( T_0 \) time constant
  \( \mu \) logarithmic decrement of attenuation
- For interested, it holds that
  \[ \frac{A_1}{A_2} = e^{\zeta \omega_n T_d}, \mu = \frac{1}{n-1} \ln \frac{A_1}{A_n} = \zeta \omega_n T_d, \omega_d = \frac{2\pi}{T_d} \]
\[ y(s) = k \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{u(\infty)}{s} \leftrightarrow y(t) = ku(\infty) \left[ 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin \left( \omega_n \sqrt{1 - \zeta^2} t + \varphi \right) \right] \]

- In limit, the bracket equals zero
- From the definition follows, that

\[ y(\infty) = ku(\infty) \rightarrow k = \frac{y(\infty)}{u(\infty)} \]

\[ T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} \rightarrow \omega_n = \frac{2\pi}{T_d \sqrt{1 - \zeta^2}} \]

- At the overshoot, the bracket reaches the maximum (i.e. \( \sin = -1 \)), so

\[ A_1 = y(t_{A_1}) - ku(\infty) = ku(\infty) \left[ 1 + \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t_{A_1}} \right] - ku(\infty) = ku(\infty) \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t_{A_1}} \]

\[ A_2 = y(t_{A_1} + T_d) - ku(\infty) = ku(\infty) \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n (t_{A_1} + T_d)} \]

- and thus

\[ \frac{A_1}{A_2} = \frac{e^{-\zeta\omega_n t_{A_1}}}{e^{-\zeta\omega_n (t_{A_1} + T_d)}} = e^{\zeta\omega_n T_d} \rightarrow \mu = \ln \frac{A_1}{A_2} = \zeta \omega_n T_d = \zeta \frac{2\pi}{\sqrt{1 - \zeta^2}} \]

\[ \mu^2 - \mu^2 \zeta^2 = 4\pi^2 \zeta^2 \rightarrow \mu^2 = \left(4\pi^2 + \mu^2\right) \zeta^2 \rightarrow \zeta = \frac{\mu}{\sqrt{4\pi^2 + \mu^2}} \]
Strejc’s identification method

- From aperiodic responses
- Find an inflex point, measure $T_u$ and $T_n$

- Calculate

$$\tau = \frac{T_u}{T_n}$$

- According its value we can approximate the response by various transfer functions

$$\tau < 0.1 \rightarrow G(s) = \frac{k}{(T_1s + 1)(T_2s + 1)}$$

$$\tau \geq 0.1 \rightarrow G(s) = \frac{k}{(Ts + 1)^n}$$
For $\tau < 0.1$, we search for a transfer function

$$G(s) = \frac{k}{(T_1s + 1)(T_2s + 1)}$$

in these steps

1) $k = \frac{y(\infty)}{u(\infty)}$

2) $t_1: y(t_1) = 0.72y(\infty)$

3) $T_1 + T_2 = \frac{t_1}{1.2564}$

4) $t_2 = 0.3574(T_1 + T_2)$

5) $y(t_2)$

6) $\tau_2 = \frac{T_1}{T_2}$

7) $T_1, T_2$
For $\tau \geq 0.1$, we search for a transfer function in these steps

1) \[ k = \frac{y(\infty)}{u(\infty)} \]

2) Normalize the step response to \[ y(\infty) = 1 \]

3) From the tangent at the inflection point determine \[ \tau = \frac{T_u}{T_n} \]

4) According to its value determine the closest higher order $n$ and the coordinate $y_i$ of the inflex point

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>0.104</td>
<td>0.218</td>
<td>0.319</td>
<td>0.41</td>
<td>0.493</td>
<td>0.57</td>
<td>0.642</td>
<td>0.709</td>
<td>0.773</td>
</tr>
<tr>
<td>$y_i$</td>
<td>0.264</td>
<td>0.327</td>
<td>0.359</td>
<td>0.371</td>
<td>0.384</td>
<td>0.394</td>
<td>0.401</td>
<td>0.407</td>
<td>0.413</td>
</tr>
</tbody>
</table>

5) From figure, determine $t_i$: \[ y(t_i) = y_i \]

6) \[ T = \frac{t_i}{n - 1} \]
**Higher order and zeros – non-oscillating case**

- Monotone smooth response (works fine for responses with „S“ shape)
  
  \[ y(t) = y(\infty) + Ae^{-\alpha t} + Be^{-\beta t} + Ce^{-\gamma t} + \cdots \]

- Subtract the steady state value and assume, that \(-\alpha\) is the slowest pole
  
  \[
  y(t) - y(\infty) \approx Ae^{-\alpha t} \\
  \ln \left( y(t) - y(\infty) \right) \approx \ln A - \alpha t \ln e \\
  \approx \ln A - \alpha t
  \]

- It is a line equation: the slope is given by \(\alpha\)
  
  axis crossing is given by \(A\)

- Place it on the figure \(\ln \left( y(t) - y(\infty) \right)\) (or \(\ln \left( y(\infty) - y(t) \right)\) for \(A < 0\))

- and determine constants \(\alpha, A\)

- Then repeat it for

  \[ y(t) - \left( y(\infty) + ae^{-\alpha t} \right) \approx Be^{-\beta t} \]

If \(y(t) - y(\infty) < 0\) (in the first step and possibly in several following steps), then \(A < 0\).

After that modify the process

\[
\ln \left( y(\infty) - y(t) \right) \approx \ln |A| - \alpha t
\]

Determine \(|A|\) and append the sign „-“.

Other details and properties are on the following pages.
Other details to the method „logarithm“

- If \( y(t) - y(\infty) < 0 \) (in the first step and possibly in several following steps), then \( A < 0 \). After that, modify the process

\[
    y(\infty) - y(t) - \approx -Ae^{-\alpha t}
\]

\[
    \ln (y(\infty) - y(t)) \approx \ln |A| - \alpha t
\]

Determine \( |A| \) and append the sign „-“

- Instead of calculating logarithms we can draw them directly on a semi-logarithmic paper – they are for \( \log_{10} \) so it is better to use common logarithm. Note that \( \log_{10} e \approx 0.4343 \)

- The method is sensitive to the adjustment of lines.
- In reasonable cases (good data with low noise), it gives a nice fitted response.
- Which does not mean, that we found the true values of time constants.

- A nice example solving „numerical“ problems can be found in the textbook Franklin-ed.6, s. 142, sekce 3.7
Identification from a frequency response

1. In the range 100-1000 rad/s the amplitude decays cca -20 dB/decade, \(|G(j300)| = -3\) dB
   estimated pole is at \(p_1 = 300\)

2. The phase rises (+180°) and \(\varphi(j2540) = 0°\)
   estimated complex pair of zeros at \(\omega_n = 2450\)

3. Amplitude slope returns to 0,
   in \(\omega = 50,000\) another pole is observed:
   This pole is at \(p_2 = 20,000\) because
   \(|G(j20,000)| = -3\) dB and phase is +45°

4. After drawing the asymptotes, we obtain
   \[
   G(s) = \frac{(s/\omega_n)^2 + 2\zeta\omega_ns + 1}{(s/p_1 + 1)(s/p_2 + 1)}
   \]

5. Difference of asymptotes and real chart

6. at „boundary frequency“ \(\omega_n = 2450\)
   is 10dB, from it follows \(\zeta = 0.16\)
In the 5th step we use the inverted chart of the resonance peak according to damping.

\[ [1 + (2\zeta/\omega_n) j\omega + (j\omega/\omega_n)^2]^{-1} \]

\[ [1 + (2\zeta/\omega_n) j\omega + (j\omega/\omega_n)^2] \]
The obtained transfer function is

\[ G(s) = \frac{(s/2450)^2 + (0.32/2450)s + 1}{(s/300 + 1)(s/20000 + 1)} = \frac{s^2 + 780s + 6000000}{s^2 + 20000s + 6000000} \]

After change of the time scale

\[ s_T = s_t \sqrt{6000000} \]
\[ \tau = t/\sqrt{6000000} \]

We obtain „nicer numbers“  \[ G(s_T) = \frac{s^2 + 0.32s + 1}{s^2 + 8.2s + 1} \]
Automatické řízení - Kybernetika a robotika

(Astrom, Murray 2008, s. 285)

- Spectral voltage analyzer (in 1s)
- Minimum → frequency of zeros
- Maximum → frequency of poles
- Good fit in the neighborhood of max. and min. → damping, multiplicity of zeros and poles

After a good fitting of the amplitude the value of time delay is found by tuning the phase plot of Bode figure

- We obtain

\[
G(s) = \frac{k \omega_2^2 \omega_3^2 \omega_5^2 (s^2 + 2\zeta_1 \omega_1 s + \omega_1^2) (s^2 + 2\zeta_4 \omega_4 s + \omega_4^2) e^{-\sigma t}}{\omega_1^2 \omega_4^2 (s^2 + 2\zeta_2 \omega_2 s + \omega_2^2) (s^2 + 2\zeta_3 \omega_3 s + \omega_3^2) (s^2 + 2\zeta_5 \omega_5 s + \omega_5^2)}
\]

kde \( \omega_k = 2\pi f_k \) a

\[ f_1 = 2.42 \text{ kHz}, \zeta_1 = 0.03, f_2 = 2.42 \text{ kHz}, \zeta_2 = 0.03, f_3 = 2.42 \text{ kHz}, \zeta_3 = 0.03, f_4 = 2.42 \text{ kHz}, \zeta_4 = 0.03, f_5 = 2.42 \text{ kHz}, \zeta_5 = 0.03 \]
Identification from Nyquist plot

- Denote \( K_\varphi = |G(j\omega_\varphi)|, \varphi = \arg G(j\omega_\varphi) \)
- Parameters \( \omega_{90}, \omega_{180}, K_0, K_90, K_{180} \) are important for control
- Gain ratio indicates the difficulty of control
  \[ \kappa = \frac{K_{180}}{K_0} = \frac{|G(\omega_{180})|}{|G(0)|} \]
- For the model we estimate parameters from

\[
G(s) = \frac{k}{1 + Ts}
\]
\[
k = K_0, T = \frac{\sqrt{\kappa^{-1} - 1}}{\omega_{180}}
\]
\[
G(s) = \frac{k}{1 + Ts} e^{-sT_d}
\]
\[
T_d = \frac{\pi - \arctan \kappa^{-2} - 1}{\omega_{180}}
\]
Example – Least squares

- For

\[ A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \]

- We minimize

\[
\min_x \sqrt{\sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} x_j - b_i \right)^2} = (2x_1 - 1)^2 + (-x_1 + x_2)^2 + (2x_2 + 1)^2
\]

- We calculate partial derivatives and put it equal to zero

\[
\frac{\partial}{\partial x_1} = 10x_1 - 2x_2 - 4 = 0, \quad \frac{\partial}{\partial x_2} = -2x_1 - 10x_2 + 4 = 0
\]

- From if follows

\[ x_1 = 1/3, \quad x_2 = -1/3 \quad ||r|| = (\frac{-1}{3})^2 + (\frac{-2}{3})^2 + (\frac{1}{3})^2 \approx 0.82 \]
Example – Least squares

For

\[ A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \]

Is

\[ x = (A^T A)^{-1} A^T b = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} \]

Then

\[ r = Ax - b = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -2/3 \\ 1/3 \end{bmatrix} \]

Thus

\[ x_1 = 1/3, \quad x_2 = -1/3 \quad \|r\| = (-1/3)^2 + (-2/3)^2 + (1/3)^2 \approx 0.82 \]
example. fit a polynomial to \( f(t) = 1/(1 + 25t^2) \) on \([-1, 1]\)

- pick \( m = n \) points \( t_i \) in \([-1, 1]\), and calculate \( y_i = 1/(1 + 25t_i^2) \)
- interpolate by solving \( Ax = b \)

\[
\begin{align*}
&n = 5 \\
&n = 15
\end{align*}
\]

(dashed line: \( f \); solid line: polynomial \( g \); circles: the points \( (t_i, y_i) \))

increasing \( n \) does not improve the overall quality of the fit
same example by approximation

- pick $m = 50$ points $t_i$ in $[-1, 1]$
- fit polynomial by minimizing $\|Ax - b\|$  

\( n = 5 \)  
\( n = 15 \)  

(dashed line: \( f \); solid line: polynomial \( g \); circles: the points \( (t_i, y_i) \))

much better fit overall
Example – Identification

measure input $u(t)$ and output $y(t)$ for $t = 0, \ldots, N$ of an unknown system

$$u(t) \xrightarrow{\text{unknown system}} y(t)$$

example ($N = 70$):

moving average model

$$y_{\text{model}}(t) = h_0 u(t) + h_1 u(t - 1) + h_2 u(t - 2) + \cdots + h_n u(t - n)$$

where $y_{\text{model}}(t)$ is the model output
Example (I/O data of page 8-15) with $n = 7$: least-squares solution is

\[
h_0 = 0.0240, \quad h_1 = 0.2819, \quad h_2 = 0.4176, \quad h_3 = 0.3536, \\
h_4 = 0.2425, \quad h_5 = 0.4873, \quad h_6 = 0.2084, \quad h_7 = 0.4412
\]
model order selection: how large should $n$ be?

- suggests using largest possible $n$ for smallest error
- much more important question: how good is the model at predicting new data (i.e., not used to calculate the model)?
model validation: test model on a new data set (from the same system)

- for $n$ too large the predictive ability of the model becomes worse!
- validation data suggest $n = 10$