7 – Steady state behaviour. Reference tracking and disturbance rejection.

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Frequency response. Transfer function.

- Stable LTI $y(s) = G(s)u(s)$ with input $u(t) = a\sin\omega t$,
  gives output (after some time, after the transients are over …)

$$y_{ss}(t) = a\left|G(j\omega)\right|\sin(\omega t + \varphi)$$

- Harmonic, changed amplitude and phase:

$$\omega \qquad \varphi = \angle G(j\omega) = \text{arctg}\frac{\text{Im}\, g(j\omega)}{\text{Re}\, g(j\omega)} \qquad \left|G(j\omega)\right|$$

- Frequency characteristic (Bode plot)

$$G(j\omega) = G(s)\big|_{s=j\omega}$$

- Transient – with system's modes
- Unstable system: definition of Bode plots is formal: $G(j\omega) = G(s)\big|_{s=j\omega}$
- Nonlinear system: Bode plots meaningless in general (output not harmonic; output is amplitude-dependent; note though equivalent transfer function, useful e.g. for auto-oscillation analysis …)

Gain. Realizability.

- Gain at ω is $|G(j\omega)|$
- DC gain (zero frequency)

$$|G(j0)| = \lim_{\omega\to 0}|G(j\omega)| = \lim_{s\to 0} G(s)$$

- Strictly proper systems (D matrix zero) block highest freqiencies:

$$G(s) = b(s)/a(s), \qquad \deg a(s) > \deg b(s) \Rightarrow \lim_{\omega\to\infty}|G(j\omega)| = 0$$

- All physical systems are strictly proper

- Bi-proper system (D matrix is nonzero):

$$\deg a(s) = \deg b(s) \Rightarrow \lim_{\omega\to\infty}|G(j\omega)| = |b_n/a_n| \neq 0, < \infty$$

- Non-proper systems amplify high frequencies with increasing gain:

$$\deg a(s) < \deg b(s) \Rightarrow \lim_{\omega\to\infty}|G(j\omega)| = \infty$$

- Not realizable (both in the “physical” sense and in the “systems” sense – no state space description

System's order. Relative order.

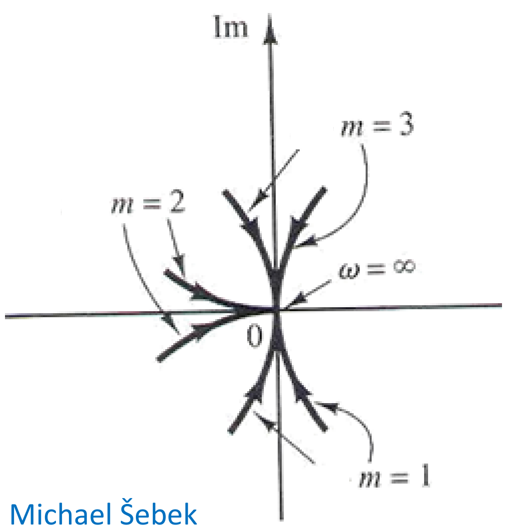
- n - # of states
- No hidden modes:

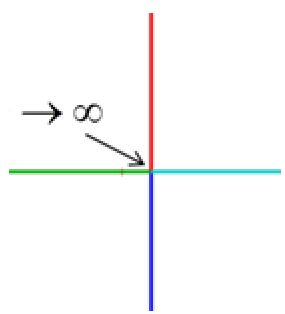
$$G(s)=b(s)/a(s),\ \deg a(s)\geq \deg b(s) \quad \Rightarrow \quad n=\deg a(s)$$

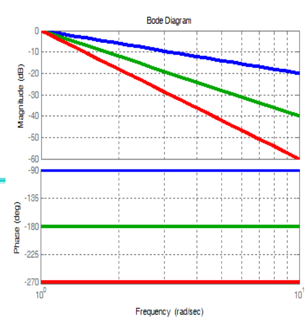
- (above holds true for proper systems)
- Relative order: $m=\deg a(s)-\deg b(s)$
- High freqs: $m\geq 1 \Rightarrow 1/s^m$
  Nyquist plot finishes ($\omega\rightarrow\infty$) at origin:

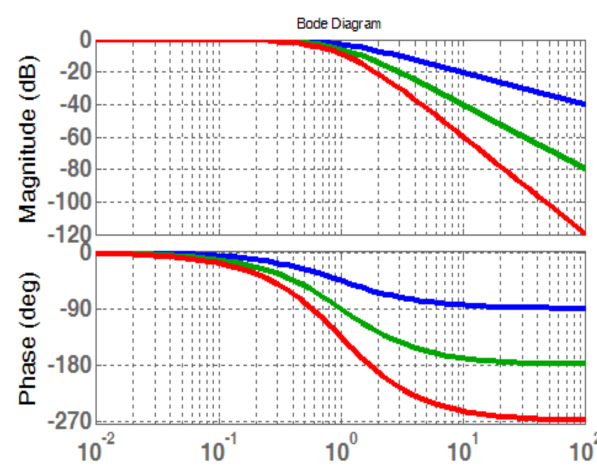
$$1/s^m : \left|1/\omega^m\right| = 1/\omega^m$$

$$\angle 1/(j\omega)^m = -m\times 90^\circ$$









Astatic systems.

- Pole(-s) at origin and its multiplicity $s = 0$

- type 0 („static“)

$$F(s) = k\frac{(s+z_1)(s+z_2)\dots}{(s+p_1)(s+p_2)\dots},\quad p_i, z_i \neq 0$$

- type 1 (astatic 1st order)

$$F(s) = k\frac{(s+z_1)(s+z_2)\dots}{s(s+p_1)(s+p_2)\dots},\quad p_i, z_i \neq 0$$

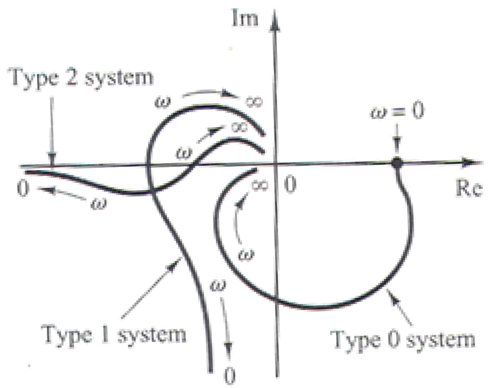
- type 2 (astatic 2nd order)

$$F(s) = k\frac{(s+z_1)(s+z_2)\dots}{s^2(s+p_1)(s+p_2)\dots},\quad p_i, z_i \neq 0$$

- etc.
- link to constant reference tracking and disturbance rejection …

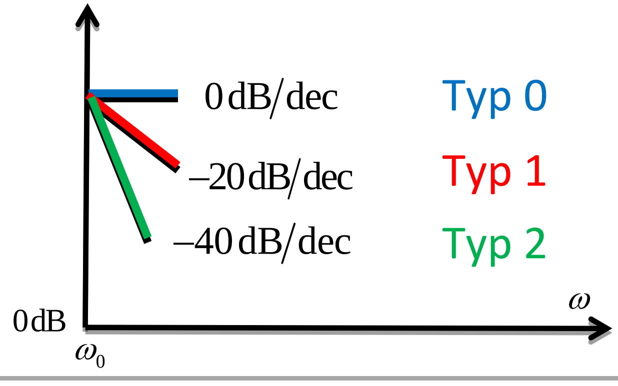
Frequency domain & astatisms

- low freqs => $l \Rightarrow k/s^l$



$$\omega \to 0 : G(j\omega) \approx k/(j\omega)^l$$

$$\lim_{\omega \to 0+} \angle g(j\omega) = -90^\circ \times l \times \operatorname{sign}(k)$$



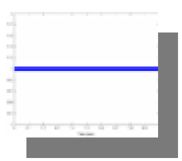
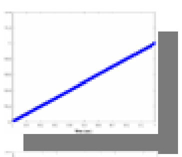
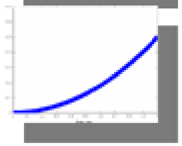
low-freqs asymptotes

Common reference command signals

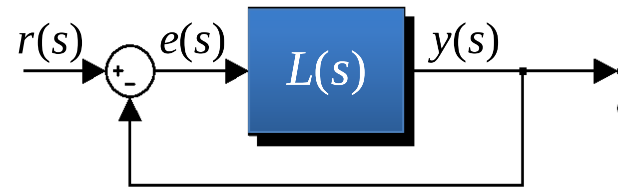
- “Steady state behaviour + transients ~ control systems quality” …
- Manipulation, reference signal tracking (servo, the tracking problem …)

- Steady-state error for $t \to \infty$

Typical reference signals used for assessment …

Shape	name	interpretation	function	L-image
• 	step	~ e.g. constant desired position	$1 \times 1(t)$	$\frac{1}{s}$
• 	rampa	~ e.g. constant desired velocity	$t \times 1(t)$	$\frac{1}{s^2}$
• 	parabola	~ e.g. constant acceleration	$\frac{1}{2} t^2 \times 1(t)$	$\frac{1}{s^3}$

Simple feedback.



All considerations apply for stable CL!
… which is not automatic at all …

- Control error

$$e(s) = S(s)r(s) = \frac{1}{1+L(s)} r(s)$$

- Steady-state value:

$$e_{ss} = \lim_{s\to 0} se(s) = \lim_{s\to 0} \frac{s}{1+L(s)} r(s)$$

- For step reference:

$$e_{\text{step,ss}} = \lim_{s\to 0} \frac{s}{1+L(s)} \frac{1}{s} = \lim_{s\to 0} \frac{1}{1+L(s)} = \frac{1}{1+\lim_{s\to 0} L(s)} = \frac{1}{1+K_p}$$

- equals zero iff. $K_p = \lim_{s\to 0} L(s) = \infty$

- i.e. open-loop (G or C) has pole at origin.

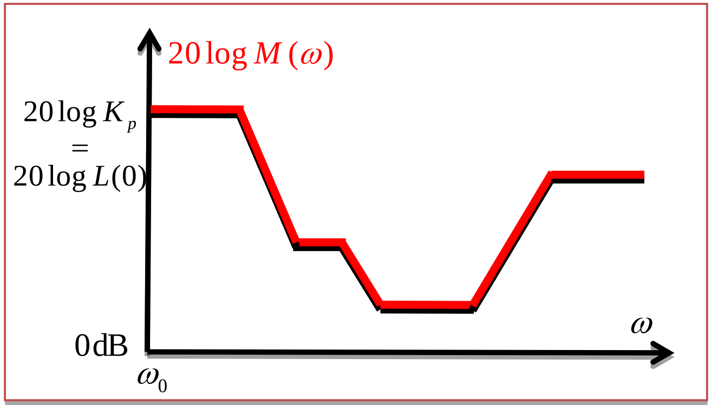
$$L(s) = \frac{(s+z_1)(s+z_2)\ldots}{s^n (s+p_1)(s+p_2)\ldots}, \quad n \geq 1$$

```
>> s = tf('s');
>> G = 1/(s+1);
>> figure, lsim(feedback(G*1,1),u,t);
>> figure, lsim(feedback(G*(1+1/s),1),u,t);
>> figure, lsim(feedback(G*(s+0.5)/(s^2+1),1),u,t);
```

- Astatic of 1st order (G or C). Note internal model principle … .
- Similar conclusions for ramp tracking, parabola tracking etc. (higher astatisms needed …)

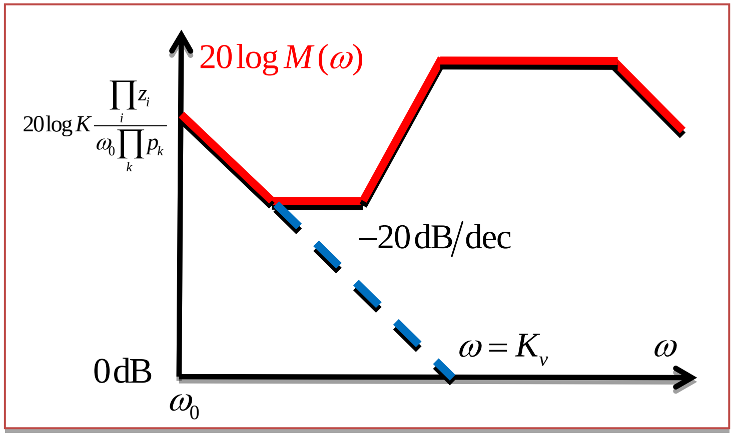
- Type 0

$$K_p = \lim_{s\to 0} L(s) \Rightarrow e_{\text{step,ss}} = \frac{1}{1+K_p}$$



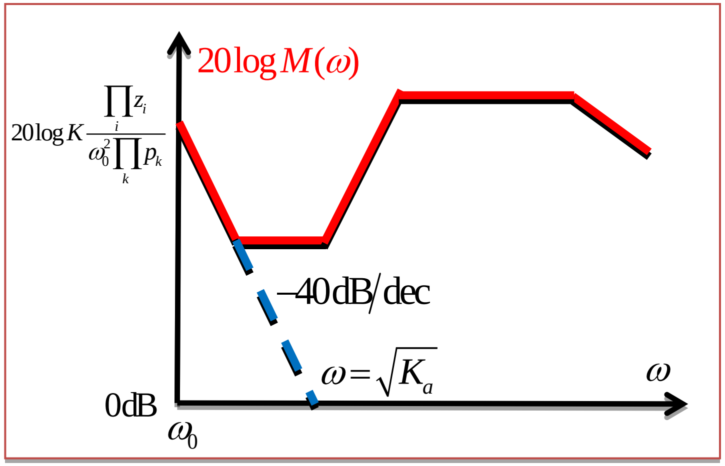
- Type 1

$$K_v = \lim_{s\to 0} sL(s) \Rightarrow e_{\text{ramp,ss}} = \frac{1}{K_v}$$



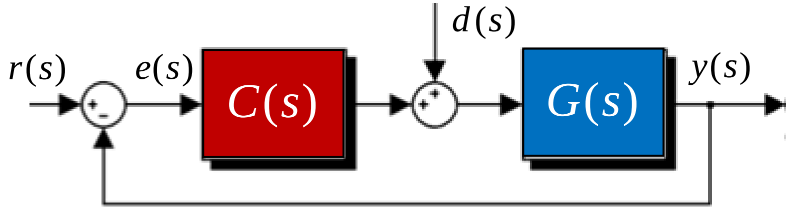
- Type 2

$$K_a = \lim_{s\to 0} s^2 L(s) \Rightarrow e_{par,ss} = \frac{1}{K_a}$$



Simple feedback. Disturbance rejection.

$$e(s) = \frac{1}{1+G(s)C(s)} r(s) - \frac{G(s)}{1+G(s)C(s)} d(s)$$



$$e_{d,ss} = \lim_{s\to 0} s e_d(s) = -\lim_{s\to 0} \frac{sG(s)}{1+G(s)C(s)} d(s)$$

$$e_d(s) = -\frac{G(s)}{1+G(s)C(s)} d(s)$$

$$d(s) = 1/s$$

$$e_d(\infty) = -\lim_{s\to 0} \frac{G(s)}{1+G(s)C(s)} = -\frac{1}{\lim_{s\to 0}\left(1/G(s)\right) + \lim_{s\to 0}\left(C(s)\right)}$$

… suggest G and C zeros/poles for step (constant) disturbance attenuation, compare to the reference tracking case …