Exercises for lectures
9 - Feedback

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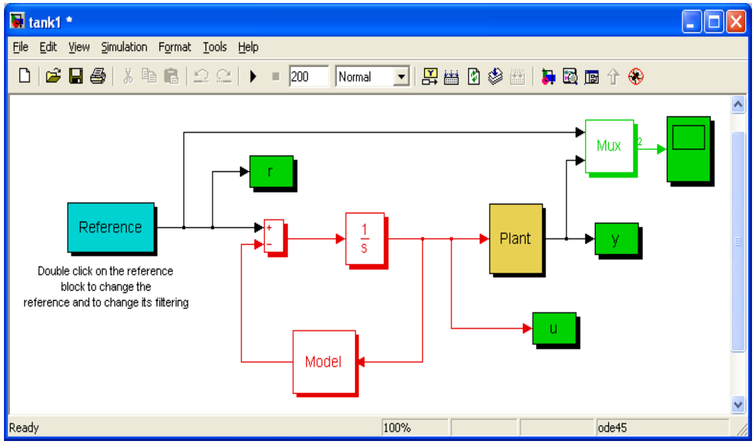
Example: Approximate inversion

- tank of a diameter 1 with water level y(t), inflow u(t) and outflow $2\sqrt{y(t)}$

$$\frac{dy(t)}{dt} + 2\sqrt{y(t)} = u(t)$$

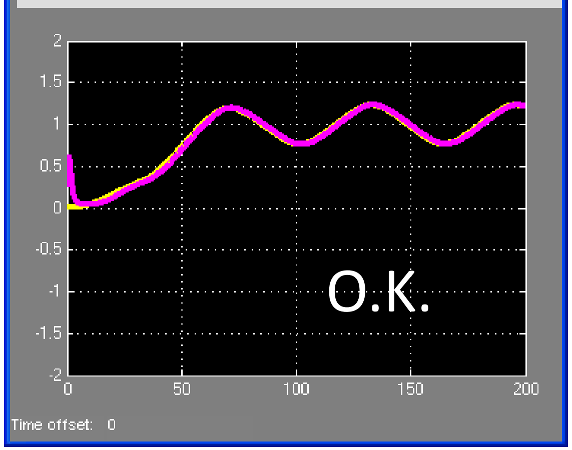
- Control goal: to track the slowly varying reference signal
- Structure: FF (feedforward) control with an approximate inversion of the model in FB (feedback) and high gain at low frequencies
- Gain is implemented by an integrator (it has ∞ gain for ω=0)
- It can be tested in Simulink – model tank1.mdl
- Woks fine only if
  - model is accurate
  - model and system have almost the same initial conditions
  - reference signal has only low freq.



Example: Approximate inversion

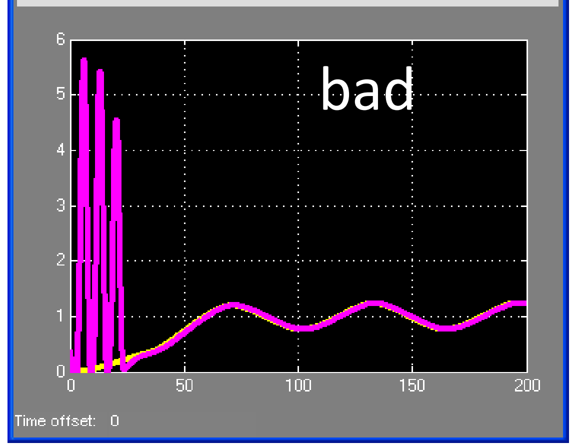
reference is slow
equal init. cond. and gain

$x_{0m} = x_{0s} = 0.25, g_m = g_s = 2$



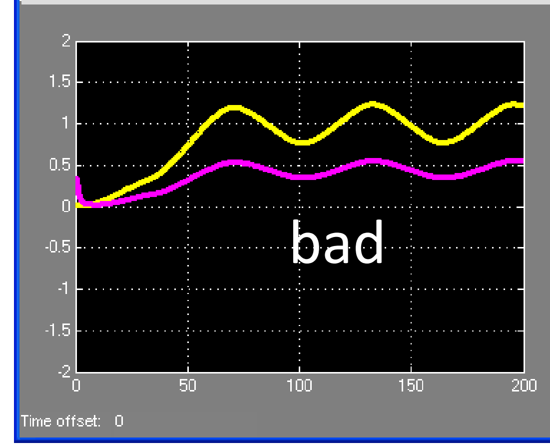
ref. is slow, different init. cond., equal gain

$x_{0m} = 4, x_{0s} = 0.25, g_m = g_s = 2$

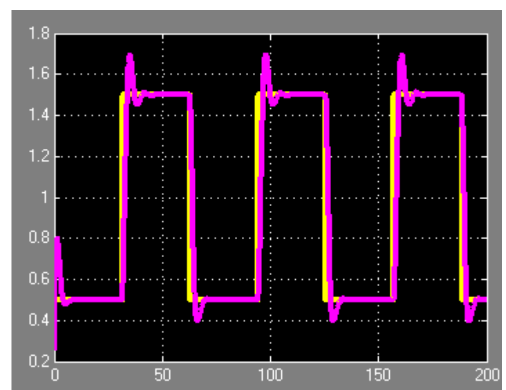


ref. is slow, same init. cond., different gain

$x_{0m} = x_{0s} = 0.25, g_m = 2, g_s = 3$

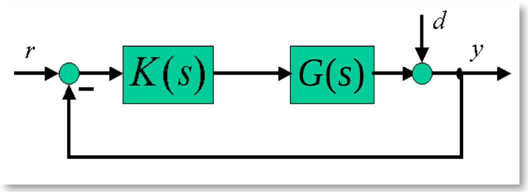


reference with high frequency (pulses)
shaping filter – lowpass
replaced only by gain 1



bad

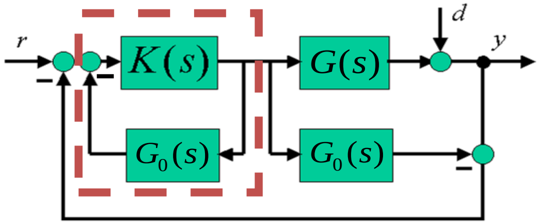
When and why to use FB?



1. To the FB circuit with the „real plant“ $G(s)$
2. artificially append its known model $G_0(s)$
3. and mark the new regulator (with plant model)

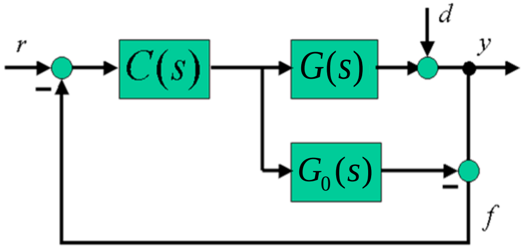
$$C(s) = \frac{K(s)}{1 + G_0(s)K(s)}$$

Note that nothing changes!



4. For the new structure, it holds that

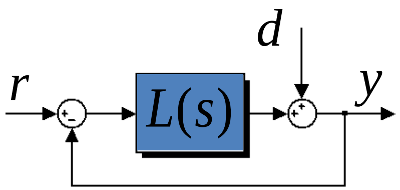
$$f = d + [G - G_0]u$$



5. The FB signal disappears $(f = 0)$ if an only if
   a) $G_0(s) = G(s)$ i.e. the plant model is known exactly
   b) $d = 0$ disturbance/initial conditions are zero

If all this is know, no FB is needed!

Why „sensitivity“ ?



Let us compare the open-loop transfer function

$$y(s) = L(s)r(s) + d(s)$$

with the closed-loop transfer function

$$y(s) = S(s)L(s)r(s) + S(s)d(s)$$

- Apparently $S(s)$ represents reduction of system sensitivity achieved with FB

In fact, this notation was firstly used by Bode for a different reason:

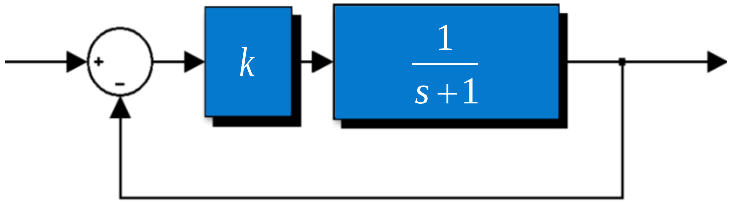
- For a scalar tran. fun., a formal derivation of T by G gives $\quad L(s) = K(s)G(s)$

$$\frac{dT}{dG} = \frac{d(GK/(1+GK))}{dG} = \frac{(1+GK)(K)-(GK)(K)}{(1+GK)^2} = \frac{K}{(1+GK)^2}$$

$$= \frac{K}{(1+GK)^2}\frac{G}{G} = \frac{GK}{1+GK}\frac{1}{1+GK}\frac{1}{G} = \frac{TS}{G} \quad \Rightarrow \quad \boxed{\frac{dT/T}{dG/G} = S}$$

- Therefore $S(s)$ is sensitivity of the relative change in CL transfer function $T(s)$ to the relative change (error) of the plant model $G(s)$

Example: Pole shift – Pizza oven acceleration



Specification: speed up 4x

- change the rise time to $T_r = 0.55$ h
- i.e. reduce the time constant to $T = 0.25$
- i.e. shift the pole from -1 to -4

$$T = \frac{T_r}{2.2}$$

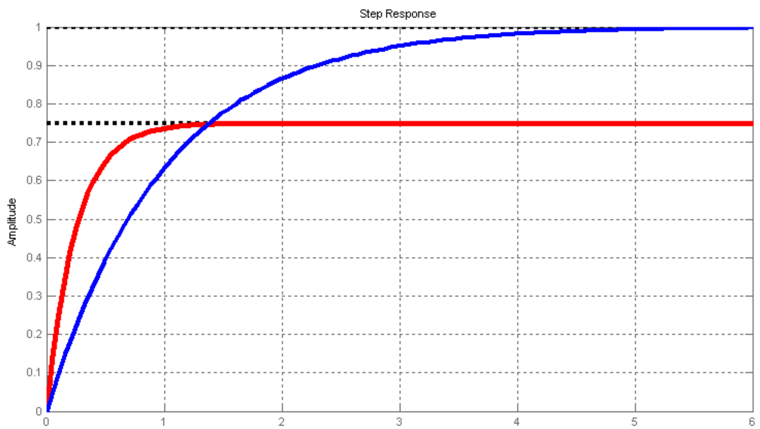
Solution

- FB + gain (P controller) – design is simple
- general CL characteristic polynomial is

$$c(s) = (s+1) + k = s + (1+k)$$

- we want to change it to $c(s) = s + 4$
- therefore let us choose $k = 3$
- We obtain the final transfer function

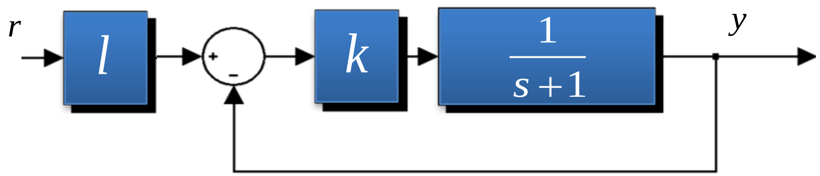
$$T(s) = \frac{L}{1+L} = \frac{3}{s+4}$$



- The response is really 4x faster,
  but what happens with the final value?

Model matching

- Better solution: to shift the pole and maintain the final steady-state value, i.e. original tran. fun. $G(s)=\frac{1}{s+1}$ changes to $F(s)=\frac{4}{s+4}$
- More complex structure is necessary



- Last time we suggested $k=3$ and obtained

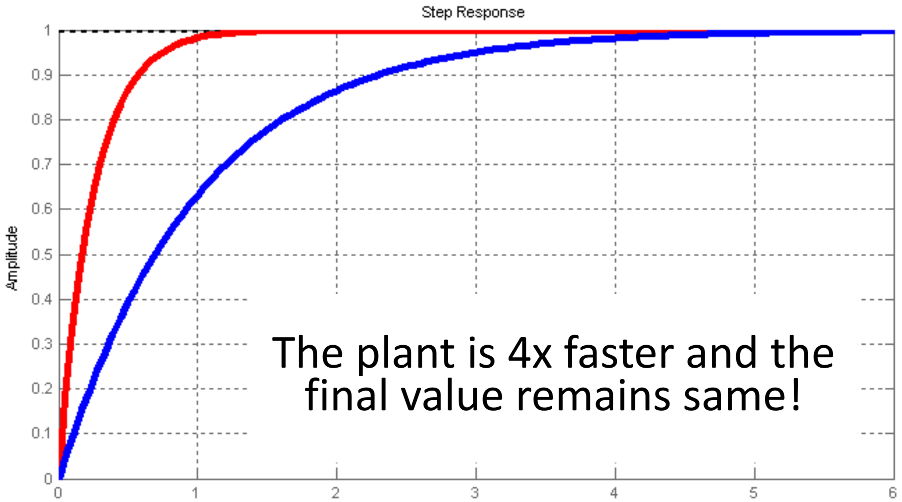
$$T(s)=\frac{3}{s+4}l$$

- Now, we just use

$$l=\frac{4}{3}$$

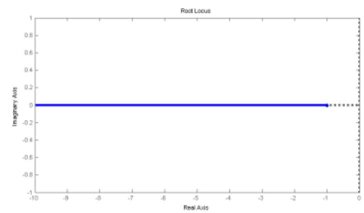
and obtain

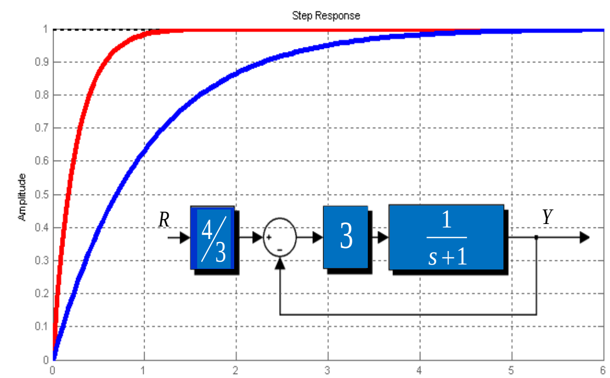
$$T(s)=\frac{4}{s+4}$$



Discussion

- We achieved the goal, but is it really so simple?
- Can we accelerate the plant arbitrarily? Can we freely move the pole?
- According the RL (root-locus) it look like yes, we can

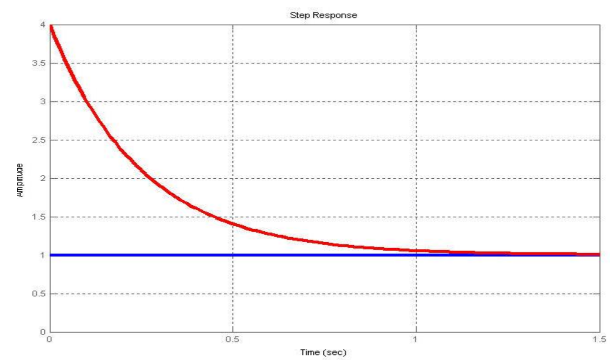




- Well, take a look at the input to the plat

$$u(s) = 4\frac{s+1}{s+4}r(s) \qquad u_{0^+} = \lim_{s\to\infty} 4\frac{s+1}{s+4}\frac{1}{s}s = 4$$

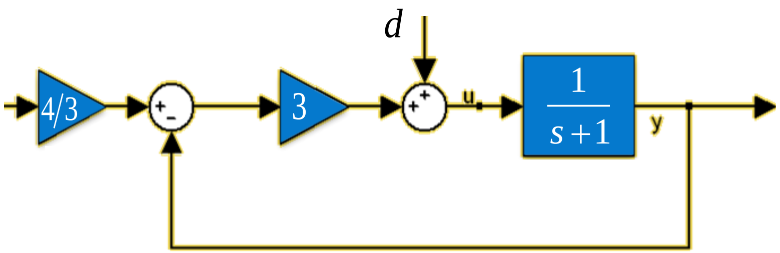
- The input signal has a high peek:
- The further we move the pole, the higher the peek will be, and the linear model won’t hold anymore.

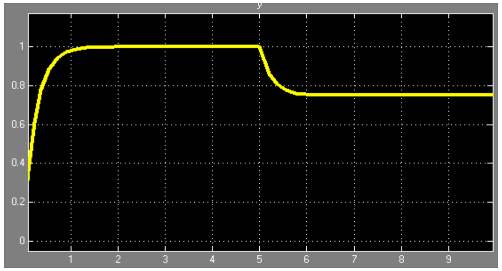


- Remark: The poles must not be moved too far from their original positions!

Discussion

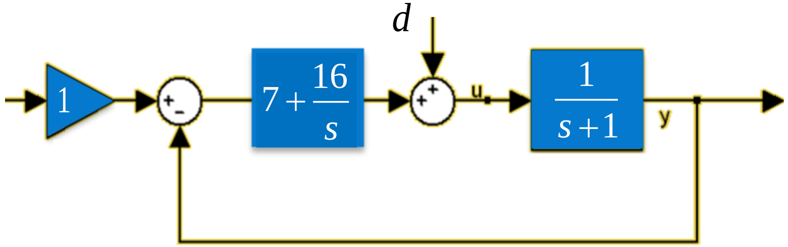
- What gives the step change in external temperature? see pizza.mdl

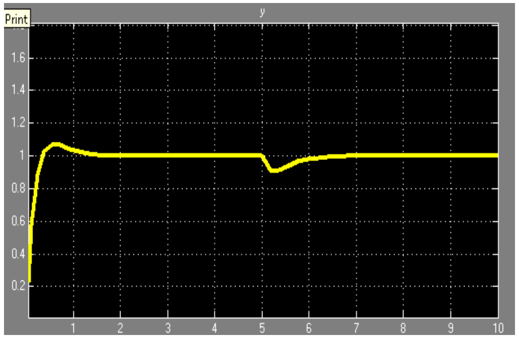




$$y(s)=\frac{4}{s+4}r(s)+\frac{1}{s+4}d(s) \quad \Rightarrow \quad y_{ss}=r_{ss}+\frac{1}{a}d_{ss}$$

- The system is unable to eliminate the step change in external temperature. The controller needs an integral term for this

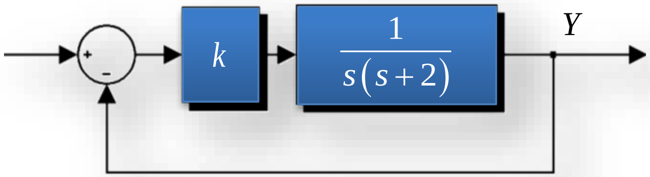


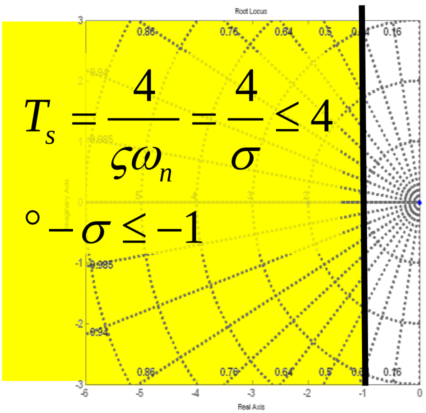


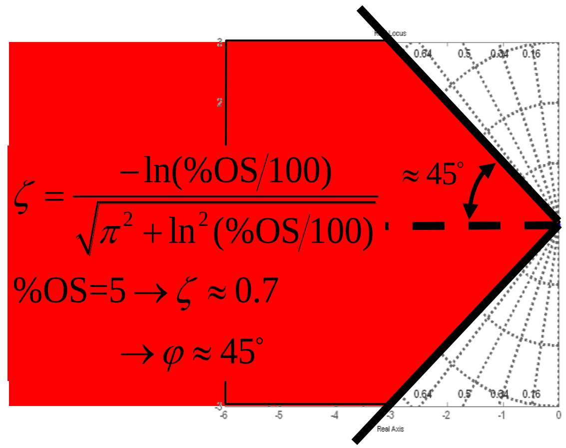
$$y(s)=\frac{7s+16}{(s+4)^2}r(s)+\frac{s}{(s+4)^2}d(s) \quad \Rightarrow \quad y_{ss}=r_{ss}+0d_{ss}$$

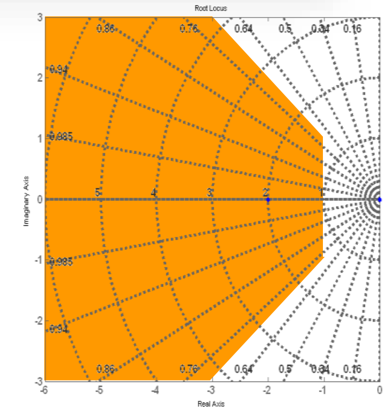
Example - 2nd order

- Design k so, that $T_s \leq 4\text{s}$ and $OS\% \leq 5\%$

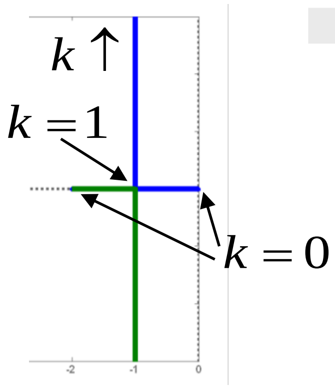


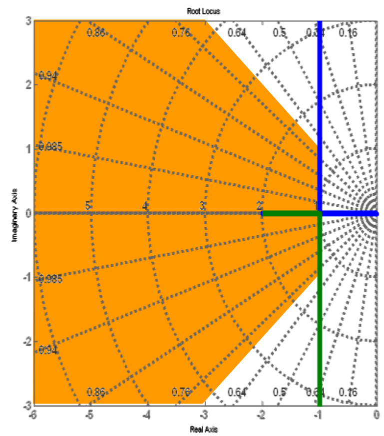


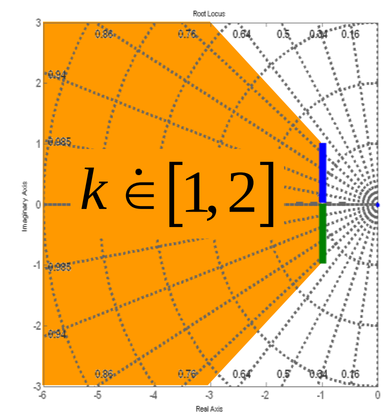




- RL







Example for 2nd and higher order: „Continuous deadbeat“

- It imitates a typical discrete strategy
- Step response is fast approaching stabilization zone and with minimal overshoot remains there.
- Typical specifications:
  1. Fast response (= minimal T_r and T_s)
  2. Overshoot between $0{,}1\%$ and 2%
  3. Undershoot $< 2\%$
  4. $E_{ss} = 0$

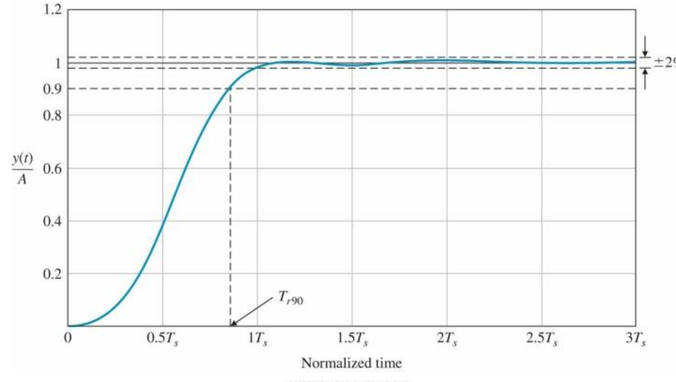


Table 10.2 Coefficients and Response Measures of a Deadbeat System

System Order	Coefficients α	β	γ	δ	ϵ	Percent Overshoot P.O.	Percent Undershoot P.U.	90% Rise Time T_{r90}	100% Rise Time T_r	Settling Time T_s
2nd	1.82					0.10%	0.00%	3.47	6.58	4.82
3rd	1.90	2.20				1.65%	1.36%	3.48	4.32	4.04
4th	2.20	3.50	2.80			0.89%	0.95%	4.16	5.29	4.81
5th	2.70	4.90	5.40	3.40		1.29%	0.37%	4.84	5.73	5.43
6th	3.15	6.50	8.70	7.55	4.05	1.63%	0.94%	5.49	6.31	6.04

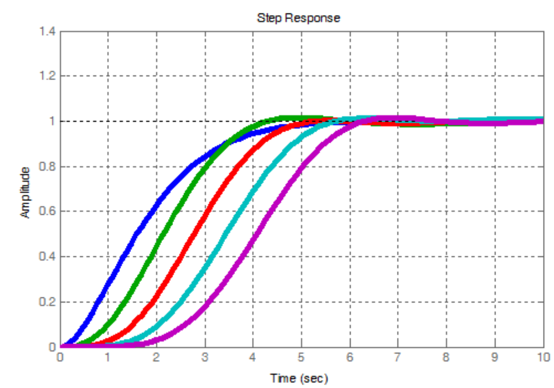
Note: All times are normalized.

- Empirically determined values for the resulting transfer functions

$$T_{2řád}(s) = \frac{\omega_n^2}{s^2 + \alpha\omega_n s + \omega_n^2} = \frac{1}{\bar{s}^2 + \alpha\bar{s} + 1}, \qquad \bar{s} = \frac{s}{\omega_n}$$

$$T_{3řád}(s) = \frac{\omega_n^3}{s^3 + \alpha\omega_n s^2 + \beta\omega_n^2 s + \omega_n^3} = \frac{1}{\bar{s}^3 + \alpha\bar{s}^2 + \beta\bar{s} + 1}$$

$$\vdots$$



Example for 2nd and higher order: „Continuous deadbeat“

- Plant with a FB controller gives closed-loop transfer function

$$G(s)=\frac{1}{s(s+1)},\quad C(s)=k\frac{s+z}{s+p} \Rightarrow T_{fb}(s)=\frac{k(s+z)}{s^3+(p+1)s^2+(p+k)s+kz}$$

- With the filter we cancel the (stable!) zero and obtain final transfer function

$$F(s)=\frac{z}{s+z} \Rightarrow T_{celk}(s)=\frac{kz}{s^3+(p+1)s^2+(p+k)s+kz}=\frac{\omega_n^3}{s^3+\alpha\omega_n s^2+\beta\omega_n^2 s+\omega_n^3}$$

- We have 1 parameter more, so let us choose T_s and calculate (using the formula for 2nd order) $\omega_n=\frac{4{,}04}{T_s}$

- Comparing the coefficients for each power

$$s^3+(p+1)s^2+(p+k)s+kz=s^3+\alpha\omega_n s^2+\beta\omega_n^2 s+\omega_n^3$$

- we obtain

$$(p+1)=\alpha\omega_n \rightarrow p=\alpha\omega_n-1$$

$$p+k=\beta\omega_n^2 \rightarrow k=\beta\omega_n^2-p$$

$$kz=\omega_n^3 \rightarrow z=\omega_n^3/k$$

Example for 2nd and higher order: „Continuous deadbeat“

- Plant, FB controller and the filter $G(s)=\dfrac{1}{s(s+1)}, C(s)=k\dfrac{s+z}{s+p}, F(s)=\dfrac{z}{s+z}$
- First, choose $T_s = 2$ s and then calculate (using the formula for 2nd order) $\omega_n = 4,04/T_s = 2,02$
- From the table, for 3rd order get $\alpha = 1,9;\ \beta = 2,2$, and compare the coefficients in $s^3+(p+1)s^2+(p+k)s+kz = s^3+\alpha\omega_n s^2+\beta\omega_n^2 s+\omega_n^3 = s^3+3,84s^2+8,98s+8,24$
- We obtain $p \approx 2,84; k \approx 6,14; z \approx 1,34$
- and we have

$$C(s)=k\frac{s+z}{s+p}=6,14\frac{s+1,34}{s+2,84}$$

$$F(s)=\frac{z}{s+z}=\frac{1,34}{s+1,34}$$

$$T_{celk}(s)=\frac{8.24}{s^3+3.84s^2+8.98s+8.24}$$

$$T_{fb}(s)=\frac{6.14s+8.24}{s^3+3.84s^2+8.98s+8.24}$$