Exercises for lectures
19 – Polynomial methods

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Division of polynomials with and without remainder

- Polynomials forms a circle, but not a body. (Circle also forms integers, in comparison to body that forms rational numbers, rational functions, etc.).
- Generally, it is not possible to divide polynomials without remainder.
- Polynomials can be divided without remainder only with so called ones, that are polynomials with 0 degree, i.e. nonzero real numbers.
- But sometimes it is possible to divide without remainder:
  - If $a(s)$ divides $c(s)$ without remainder, we denote it $a(s)\,|\,c(s)$
    Then exists such $b(s)$, that $c(s)=a(s)b(s)$
  - Then $a(s)$ is divisor of $c(s)$, and $c(s)$ is a multiple of $a(s)$
  - If $a(s)$ divides $c(s)$, $d(s)$, then $a(s)$ is a common divisor of $c(s)$, $d(s)$
  - The greatest common divisor is the one with the highest degree
  - Example

    For $c(s)=(s+1)(s-1)$ is $a(s)\,|\,c(s)$

    $a(s)=s+1$

    $d(s)=s+2$ $\qquad d(s)\nmid c(s)$

Greatest common divisor

- greatest (left) common divisor = největší společný dělitel
- For

$$g(s) = \gcd\left(a(s), b(s)\right)$$

is

$$a(s)p(s) + b(s)q(s) = g(s)$$
$$a(s)v(s) + b(s)w(s) = 0$$

and the matrix

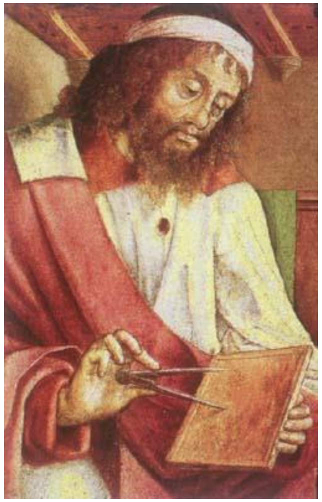
$$\mathbf{U}(s) = \begin{bmatrix} p(s) & q(s) \\ v(s) & w(s) \end{bmatrix}$$

is unimodular (= its determinant is non-zero constant)

```
>> pformat rootr
>> a=(s+1)^2*(s-1)*(s+2)
a =
 (s+2)(s^2+2s+1)(s-1)
>> b=(s+1)*(s-1)*(s-2)
b =
       (s+1)(s-1)(s-2)
>> g=grd(a,b)
g =
       (s+1)(s-1)
>> [g,U]=grd(a,b)
g =
       (s+1)(s-1)
U =
   0.0833          -0.0833(s+5.0000)
  -0.2294(s-2)     0.2294(s+2)(s+1)
>> U*[a; b]
ans =
       (s+1)(s-1)
         0
```

Division with remainder

- Euklid of Alexandria
  (~ 300 BCE)
- introduced division with remainder
- For given polynomials $a(s)$, $b(s) \neq 0$
  there exist polynomials $q(s)$, $r(s)$, such that



$$a = b\boxed{q} + \boxed{r}, \quad \deg r < \deg b$$

quotient remainder

- Therefore polynomials form so-called Euclidean circle.

```
>> a=prand(5,'int'),b=prand(3,'int')
a = -8 - 7s + 3s^2 - 2s^3 + 3s^4 + 4s^5
b =  6 + 3s + 6s^2 - 6s^3
>> [q,r] = rdiv(a,b)
q = -1.2 - 1.2s - 0.67s^2
r = -1 + 3.5s + 18s^2
>> a-(b*q+r)
ans = 0
```

A necessary and sufficient condition for solvability

Diophantus of Alexandria

Greek mathematician

3. century A.D.

Lemma: Equation

$$a(s)x(s)+b(s)y(s)=c(s)$$

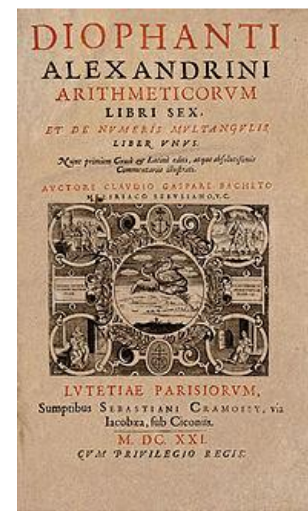
has solution if and only if $\gcd\left(a,b\right)|\,c$

Proof

Necessity („only if“):

Let $ax'+by'=c$ and denote $\gcd(a,b)=g,\ a=g\bar{a},\ b=g\bar{b}$

Then $g(\bar{a}x'+\bar{b}y')=c$ and hence $g\,|\,c$



Sufficiency („if“)

Let $(a,b)\,|\,c$ and denote $(a,b)=g,\ c=g\bar{c}$

Then always exists p, q, such that $ap+bq=g$

Multiplying by $\bar{c}$ we obtain $a(p\bar{c})+b(q\bar{c})=c$

and we obtained the solution $x=p\bar{c},\,y=q\bar{c}$

Example: Solvability

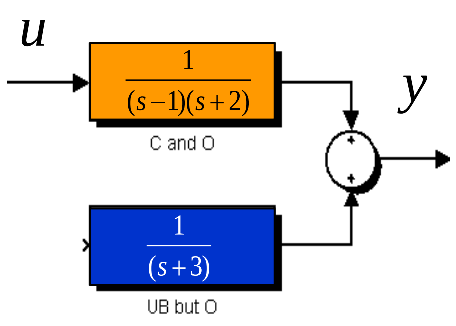
```
>> c=(s+1)*(s+2);
>> a=(s+1)^2*(s-1); b=(s+1)*(s-2);
>> g=gld(a,b)
 g = (s+1)
>> pol(c/g)
 ans = (s+2.0000)
>> [x,y]=axbyc(a,b,c)
 x =  1.3333
 y = -1.3333(s+1.2500)
>> a*x+b*y-c
 ans = 0
```

```
>> c=(s-1)*(s+2)
c = (s+2)(s-1)
>> c/g
ans = (s+2)(s-1)/(s+1)
>> pol(c/g)
??? Error using ==> frac.pol
Argument is not convertible
to polynomial.
```

Example: interpretation of conditions of solvability

- Transfer function without hidden modes (no cancellation)

$$y(s)=\frac{1}{(s-1)(s+2)}\frac{(s+3)}{(s+3)}u(s)$$



$$y(s)=\frac{b(s)}{a(s)}u(s)\qquad \begin{aligned}a(s)&=(s-1)(s+2)(s+3)\\ b(s)&=(s+3)\end{aligned}$$

- Characteristic polynomial is

$$a(s)x(s)+b(s)y(s)=(s-1)(s+2)(s+3)x(s)+(s+3)y(s)=(s+3)d(s)$$

- So no FB regulator changes uncontrollable part

```
>> pformat rootr
>> a=(s-1)*(s+2)*(s+3),b=s+3
a = (s+3.0000)(s+2.0000)(s-1.0000), b = (s+3)
>> x=prand(2);y=prand(2);c=a*x+b*y
c = 0.2877(s+3.0629)(s+3.0000)(s+1.6799)(s^2-3.3072s+3.0550)
>> x=prand(2);y=prand(2);c=a*x+b*y
c = 0.7258(s+3.0000)(s+1.4780)(s+1.1711)(s^2-1.9064s+1.4597)
>> x=prand(2);y=prand(2);c=a*x+b*y
c = -0.8323(s+3.0000)(s+2.4112)(s+0.8026)(s-0.6043)(s-1.4946)
```

General solution

Lemma: General solution

- General solution has the form $\boxed{\begin{aligned} x &= x' - \bar{b}t \\ y &= y' + \bar{a}t \end{aligned}}$ where t is a random polynomial parameter

Proof:

1) It's a solution for every t: Simply substitute it into equation

$$ax + by = ax' - a\bar{b}t + by' + b\bar{a}t = ax' + by' + \left(b\bar{a} - a\bar{b}\right)t = c$$

2) There is no other solution:

- For any solution x, y and x', y' it holds, that $ax + by = c, \quad ax' + by' = c$
- Subtracting $a(x - x') + b(y - y') = 0$ and from it $-a(x - x') = (y - y')b$
- Previously defined polynomials $\bar{a}, \bar{b}$ are relatively prime and $a\bar{b} = \bar{a}b$
- Thus $\begin{aligned} \bar{b} &\mid x - x' \\ \bar{a} &\mid y - y' \end{aligned}$, hence for some polynomial t is $\begin{aligned} x - x' &= -\bar{b}t \\ y - y' &= \bar{a}t \end{aligned}$
- Any such solution of t we obtain by going through the set of all possible polynomials

Example: General solution

- Some solution

```
a=(s+1)^2*(s-1)
a = -1 - s + s^2 + s^3
>> b=(s+1)*(s-2)
b = -2 - s + s^2
>> c=(s+1)*(s+2)
c =  2 + 3s + s^2
>> [x,y]=axbyc(a,b,c)
x =  1.3333
y = -1.7 - 1.3s
>> [x,y,v,w]=axbyc(a,b,c)
x =  1.3333
y = -1.7 - 1.3s
v =  0.76 - 0.38s
w = -0.38 + 0.38s^2
```

- Other solution

```
>> t=1-s
t = 1 - s
>> xnew=x+v*t,ynew=y+w*t
xnew =
   2.1 - 1.1s + 0.38s^2
ynew =
  -2 - 0.96s + 0.38s^2 - 0.38s^3
>> a*xnew+b*ynew-c
ans =
     0
```

$$x_{new}(s) = x'(s) + r(s)t(s)$$
$$y_{new}(s) = y'(s) + v(s)t(s)$$

$$r(s) = -\bar{b}(s)$$
$$v(s) = \bar{a}(s)$$

$$x(s) = x'(s) - \bar{b}(s)t(s)$$
$$y(s) = y'(s) + \bar{a}(s)t(s)$$

Solution of minimum degree

- We take a general solution

$$x = x' - \bar{b}t$$
$$y = y' + \bar{a}t$$

by a division algorithm we reduce x' modulo $\bar{b}$: $\begin{array}{l} x' = \bar{b}q + r \\ \deg r < \deg \bar{b} \end{array}$

- The it holds that

$$x = r - \bar{b}(t - q)$$
$$y = y' + \bar{a}t$$

- By choice $t = q$ we obtain a solution x, y of minimum degree in x.

$$\begin{aligned} x &= r \\ y &= y' + \bar{a}q \end{aligned} \qquad \deg x < \deg \bar{b}$$

- Similarly, we prove the existence and unicity of solution of minimum degree in y.

- These two solutions are generally different.

Example: Solution of minimum degree

- Data

```
>> a=prand(3),b=prand(2),c=prand(5)
a = 0.62 - s + 1.5s^2 + 0.43s^3
b = 0.47 + 1.3s + 0.64s^2
c = 1.3 - 0.91s - 2.3s^2 + 1.8s^3 + 0.39s^4 + 0.02s^5
>> [x,y,b_bar,a_bar]=axbyc(a,b,c); b_bar,a_bar
b_bar = -0.19 - 0.51s - 0.25s^2
a_bar =  0.25 - 0.42s + 0.61s^2 + 0.17s^3
```

- Solution of min. degree y

```
>> [x,y]=axbyc(a,b,c)
x = 1.9 + 2.4s + 0.047s^2
y = 0.34 - 1.9s - 1.1s^2
>> [x,y]=axbyc(a,b,c,'miny')
x = 1.9 + 2.4s + 0.047s^2
y = 0.34 - 1.9s - 1.1s^2
```

- Solution of min. degree x

```
>> [x,y]=axbyc(a,b,c,'minx')
x = 1.8 + 2.3s
y = 0.38 - 2s - s^2 + 0.032s^3
```

Coincidence

- An important special case occurs when $\deg c < \deg a + \deg b$

  Explanation (we expect relatively prime a, b)

$$ax + by = c \;\mapsto\; \boxed{\frac{x}{b}} + \boxed{\frac{y}{a}} = \frac{c}{ab}$$

strictly proper, if it holds

strictly proper, if sol. min deg x

strictly proper, if sol. min deg y

  - Right side strictly proper ➡ either both fractions on the left are strictly proper or neither one is
  - Right side not strictly proper ➡ only one fraction on the left may be strictly proper
- If $\deg c < \deg a + \deg b$, then both solutions of minimum degree coincide and there is only one solution of minimum degree (which is minimal in both unknowns at the same time).
- If $\deg c \geq \deg a + \deg b$, then there exist two different solutions of minimum degree (one in x and the other one in y).

Example: Coincidence

Yes.

- Both are the same!
- So there is only one solution of minimum degree.

```
>> a=prand(3),b=prand(2),c1=1, c2=prand(6)
a = -1.5 + 0.22s - 1.4s^2 - 0.84s^3
b =  0.76 + 0.38s - 1.3s^2
c1 = 1
>> [x,y]=axbyc(a,b,c1,'minx')
x = -0.43 + 0.11s
y =  0.45 + 0.13s - 0.071s^2
>> [x,y]=axbyc(a,b,c1,'miny')
x = -0.43 + 0.11s
y =  0.45 + 0.13s - 0.071s^2
```

No.

- They are different.

```
>> c2=prand(6)
c2 = -0.71+0.51s-0.42s^2+0.23s^3
               -0.96s^4-0.15s^5+0.74s^6
>> [x,y]=axbyc(a,b,c2,'minx')
x = 0.49 - 0.28s
y = 0.064 - 0.064s + 0.56s^2
               - 0.046s^3 - 0.55s^4
>> [x,y]=axbyc(a,b,c2,'miny')
x = -0.28 - 0.16s + 1.6s^2 - 0.89s^3
y = -1.5 + 1.2s - 0.99s^2
```

Elementary operations on polynomial matrix

- Row operations - 3 basic ones
  - row multiplication by a non-zero constant
  - row interchange
  - row addition of multiplied polynomial to other row

$$\begin{bmatrix} 1 & s \\ 2 & s^2 \end{bmatrix} \xrightarrow{\text{1. row } \times 3} \begin{bmatrix} 3 & 3s \\ 2 & s^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & s \\ 2 & s^2 \end{bmatrix} \xrightarrow{\text{row switch}} \begin{bmatrix} 2 & s^2 \\ 1 & s \end{bmatrix}$$

$$\begin{bmatrix} 1 & s \\ 2 & s^2 \end{bmatrix} \xrightarrow{\text{1. row + s}\times\text{ 2.ord.}} \begin{bmatrix} 1+2s & s+s^3 \\ 2 & s^2 \end{bmatrix}$$

- Column operations are dual.
- Elementary operations, except the multiplications, maintain the determinant.
- They correspond to multiplication by unimodular matrices (i.e. matrix with a constant non-zero determinant).

Solution by a polynomial reduction

Solution of equations by polynomial reductions

Step 1 Form a composite matrix

$$\begin{bmatrix} a(s) & 1 & 0 \\ b(s) & 0 & 1 \end{bmatrix}$$

Step 2 Reduce it by elementary operations to the form

$$\begin{bmatrix} g(s) & p(s) & q(s) \\ 0 & v(s) & w(s) \end{bmatrix}$$

Then

$$p(s)a(s)+q(s)b(s)=g(s)$$
$$v(s)a(s)+w(s)b(s)=0$$

where

$$\gcd\big(a(s),b(s)\big)=g(s)$$
$$\gcd\big(v(s),w(s)\big)=1$$

Step 3 Extract $g(s)$ z $c(s)$ and obtain $c(s)=\bar{c}(s)g(s)$

If it doesn’t work, the equation doesn’t have a solution!

Solution by a polynomial reduction

The result: as the solution take

$$x(s) = \bar{c}(s)p(s)$$
$$y(s) = \bar{c}(s)q(s)$$

Moreover, all solutions are expressed as follows

$$x(s) = \bar{c}(s)p(s) + v(s)t(s)$$
$$y(s) = \bar{c}(s)q(s) + w(s)t(s)$$

free polynomial parameter

The procedure of calculations follows from the equality

$$\begin{bmatrix} p & q \\ v & w \end{bmatrix} \begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} = \begin{bmatrix} g & p & q \\ 0 & v & w \end{bmatrix}$$

Example: Solution by a polynomial reduction

$$(s+1)x(s)+(s-1)y(s)=s$$

Step 1 and 2

$$\left[\begin{array}{c:cc} s+1 & 1 & 0 \\ s-1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{c:cc} s+1 & 1 & 0 \\ -2 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{c:cc} 1 & ½ & -½ \\ s+1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{c:cc} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1-s}{2} & \frac{1+s}{2} \end{array}\right]$$

Step 3

$$g(s)=1 \Rightarrow \bar{c}(s)=s$$

Step 4

$$x(s)=\frac{s}{2} \qquad x(s)=\frac{s}{2}+\frac{1-s}{2}t(s)$$

$$y(s)=-\frac{s}{2} \qquad y(s)=-\frac{s}{2}+\frac{1+s}{2}t(s)$$

Solution by a Sylvester matrix

Example of 2nd order polynomial,
where we search for $x(s) = x_0 + x_1 s,\ \ y(s) = y_0 + y_1 s$

$$a(s) = a_0 + a_1 s + a_2 s^2$$
$$b(s) = b_0 + b_1 s + b_2 s^2$$
$$c(s) = c_0 + c_1 s + c_2 s^2$$

Step 1: Substituting polynomials with unknow coefficients into equation

$$\left(a_0 + a_1 s + a_2 s^2\right)\left(x_0 + x_1 s\right) + \left(b_0 + b_1 s + b_2 s^2\right)\left(y_0 + y_1 s\right) = c_0 + c_1 s + c_2 s^2$$

Comparing coefficients of identical powers,
or in matrix form

$$\begin{aligned} a_0 x_0 + b_0 y_0 &= c_0 \\ a_1 x_0 + b_1 y_0 + a_0 x_1 + b_0 y_1 &= c_1 \\ a_2 x_0 + b_2 y_0 + a_1 x_1 + b_1 y_1 &= c_2 \\ a_2 x_1 + b_2 y_1 &= 0 \end{aligned}$$

$$\begin{bmatrix} x_0 & y_0 & x_1 & y_1 \end{bmatrix} \boxed{\begin{bmatrix} a_0 & a_1 & a_2 & 0 \\ b_0 & b_1 & b_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \end{bmatrix}} = \begin{bmatrix} c_0 & c_1 & c_2 & 0 \end{bmatrix}$$

We will solve this matrix equation, and obtain x_0, y_0, x_1, y_1. Then we compose the searched result. $x(s) = x_0 + x_1 s,\ \ y(s) = y_0 + y_1 s$

Example: Solution by a Sylvester matrix

$$(s+1)x(s)+(s-1)y(s)=s$$

$$a(s)=1+s \quad b(s)=-1+s \quad c(s)=s$$

$$x(s)=x_0 \quad y(s)=y_0$$

$$\begin{bmatrix} x_0 & y_0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_0 & y_0 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$x(s)=\frac{1}{2},\, y(s)=\frac{1}{2}$$

- We got the minimum degree solution (in both unknowns), other than that particular solution obtained earlier.

- The solution from the last example is obtained from the general solution by choosing $t(s) = 1$

$$x(s)=\frac{s}{2}+\frac{1-s}{2}t(s)$$

$$y(s)=-\frac{s}{2}+\frac{1+s}{2}t(s)$$

Beware of miscalculation

- For equation

$$a = 1 + s^2, b = (1+s)^2 = 1 + 2s + s^2, c = 1 + 2s + 2s^2$$

we look for a solution of 0 degree (knowing that it might be wrong) and try to solve the matrix equation

$$\begin{bmatrix} x_0 & y_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$$

that has obviously no solution.

- Still the polynomial equation has a solution, of higher degrees, e.g.

$$x = -0.5s, y = 1 + 0.5s$$

- This is a typical example

```
>> a=1+s^2,b=(1+s)^2,c=a+b-1
a =  1 + s^2
b =  1 + 2s + s^2
c =  1 + 2s + 2s^2
>> S=sylv([a;b],0),C=c{0:2}
S =  1     0     1
     1     2     1
C =  1     2     2
>> XY=C/S
XY = 0.5000    1.0000
>> XY*S==C
ans = 0     0     0
>> [x,y]=axbyc(a,b,c)
x =  -0.5s
y =   1 + 0.5s
```

Example: pole assignment

System – motor

Input voltage transfer function

- Previously designed PI controller gives a zero error to step but not a good dynamics

- Lets try better *c*

```
b =  0.0670
a =  0.017 + 0.0079s + 0.0011s^2
p = s
q = 15 + 3s
c1 = a*p + b*q
c1 = 1 + 0.22s + 0.0079s^2 + 0.0011s^3
>> roots(c1)
ans =
  -1.1476 +13.6253i
  -1.1476 -13.6253i
  -4.8866 + 0.0000i


>> c2=(s+5)*(s+1+j)*(s+1-j)
c2 =
     10 + 12s + 7s^2 + s^3
```

Example: pole assignment

```
>> [x1,y1]=axbyc(a,b,c2)
x = -1.7e+02 + 9.1e+02s
y =  1.9e+02 - 32s
```

- The solution is a general regulator of 1st order.
- It ensure good dynamic, but it doesn’t have an integrator and therefore it doesn’t ensure zero error.
- Lets try to force an integrator in the dynamics. (solving the equation we will make part of the system)
- We get a controller with good dynamics and zero steady-state error
- but it is a PID, what we could expect.

```
>> [x2,y2]=axbyc(a*s,b,c2)
x2 = 909.0909
y2 = 1.5e+02 - 52s - 2.7s^2
>> p2 = x2*s, q2 =y2
p2 = 9.1e+02s
q2 = 1.5e+02 - 52s - 2.7s^2
```

Remarks:

- we have good control over dynamics (by appropriate selection of CL poles)
- we can ensure additional requirements
- but we have no control over the regulator order - it just comes out

Example: Proper controller

Example:

- When the right side of the equation doesn't have a sufficiently high degree.
- There is only a small chance for a proper controller.
  (It is not a generic case.)

```
>> a=(s-1)^2,b=s,c=(s+1)^2
a = 1 - 2s + s^2
b = s
c = 1 + 2s + s^2
>> [x,y]=axbyc(a,b,c)
x =  1.0000
y =  4.0000
```

Other example:

- Proper regulator exists.
  This is a generic case.

```
>> c=prand(2,'sta')
c = 0.86 + 2.6s + s^2
>> [x,y]=axbyc(a,b,c)
x = 0.8573
y = 4.4 + 0.14s
```

All stabilizing controllers - explicitly

- We assume that $a(s), b(s)$ are relatively prime

Explicit parameterization of all stabilizing controllers

- If $c(s)$ is some stable polynomial and $x(s), y(s)$ is some solution of

$$a(s)x(s)+b(s)y(s)=c(s)$$

- then all stabilizing controllers are given by

$$\boxed{\frac{q(s)}{p(s)}=\frac{y(s)d(s)-a(s)t(s)}{x(s)d(s)+b(s)t(s)}}$$

- $t(s)$ is a random polynomial,
- $d(s)$ is a random stable pol.

- Part of the parameterization are all possible reduction in the controller transfer function.

Proper controller is obtained under following conditions:

- Proper system t. f. $b(s)/a(s)$ and $\deg c(s)=2\deg a(s)-1$
- Proper fraction $t(s)/d(s)$ and $p(s)\not\equiv 0$

- What is the form of a characteristic polynomial?
- For $t(s)=0, d(s)=1$

$$\frac{q(s)}{p(s)}=\frac{y(s)}{x(s)} \Rightarrow a(s)p(s)+b(s)q(s)=a(s)x(s)+b(s)y(s)=c(s)$$

- For $t(s)$ random, $d(s)$ random stable (no „reduction in cont.“)

$$\frac{q(s)}{p(s)}=\frac{y(s)d(s)-a(s)t(s)}{x(s)d(s)+b(s)t(s)} \Rightarrow$$

$$\begin{aligned} a(s)p(s)+b(s)q(s) &= a(s)\big(x(s)d(s)+b(s)t(s)\big)+b(s)\big(y(s)d(s)-a(s)t(s)\big) \\ &= \big(a(s)x(s)+b(s)y(s)\big)d(s)+\big(a(s)b(s)-b(s)a(s)\big)t(s)=c(s)d(s) \end{aligned}$$

- Is there always factor $c(s)$? In this case it would not be a general stable characteristic polynomial?
- No, it is not there:
  In some cases, it gets cancelled "in the controller“ already.

Cancellation (reduction)

- For parameters, that lead to cancellation (reduction)

$$\frac{q(s)}{p(s)}=\frac{\left(y(s)d(s)-a(s)t(s)\right)/c(s)}{\left(x(s)d(s)+b(s)t(s)\right)/c(s)} \Rightarrow$$

$$a(s)p(s)+b(s)q(s)=a(s)\frac{\left(y(s)d(s)-a(s)t(s)\right)}{c(s)}+b(s)\frac{\left(x(s)d(s)+b(s)t(s)\right)}{c(s)}$$

$$=\frac{\left(a(s)x(s)+b(s)y(s)\right)d(s)+\left(a(s)b(s)-b(s)a(s)\right)t(s)}{c(s)}=\frac{c(s)d(s)}{c(s)}=d(s)$$

- The resulting characteristic polynomial can be any stable pol.
- When does it happens? It has to be:

$$y(s)d(s)-a(s)t(s)=c(s)v(s)$$
$$x(s)d(s)+b(s)t(s)=c(s)w(s)$$

- We solve one equation, the second „comes out“.

$$y(s)d(s)=c(s)v(s)+a(s)t(s)$$
$$x(s)d(s)=c(s)w(s)-b(s)t(s)$$

- Final controller („after reduction“)

$$\frac{q(s)}{p(s)}=\frac{v(s)}{w(s)}$$

Example

- System $b(s)/a(s) = 1/s$
- is stabilized by a controller $y_1(s)/x_1(s) = 2/1 = 2$
  with resulting char. polynomial $a(s)x_1(s) + b(s)y_1(s) = s + 2$
- and also controller $y_2(s)/x_2(s) = 1/1 = 1$
  with resulting char. polynomial $a(s)x_2(s) + b(s)y_2(s) = s + 1$
- Consider the first controller
  and write the parametrization in the form $\dfrac{q(s)}{p(s)} = \dfrac{2d(s) - st(s)}{d(s) + t(s)}$
- We choose parameters so that the "first char. pol.“ is cancelled
- We solve the equation $y_1(s)d(s) = c(s)v(s) + a(s)t(s)$ $\Rightarrow$

$$\Rightarrow \quad 2(s+1) = (s+2)v(s) + st(s) \quad \Rightarrow \quad v(s) = t(s) = 1$$

- After substitution
  and we obtain $\dfrac{q(s)}{p(s)} = \dfrac{2(s+1) - s}{(s+1)+1} = \dfrac{s+2}{s+2} = \dfrac{1}{1}$

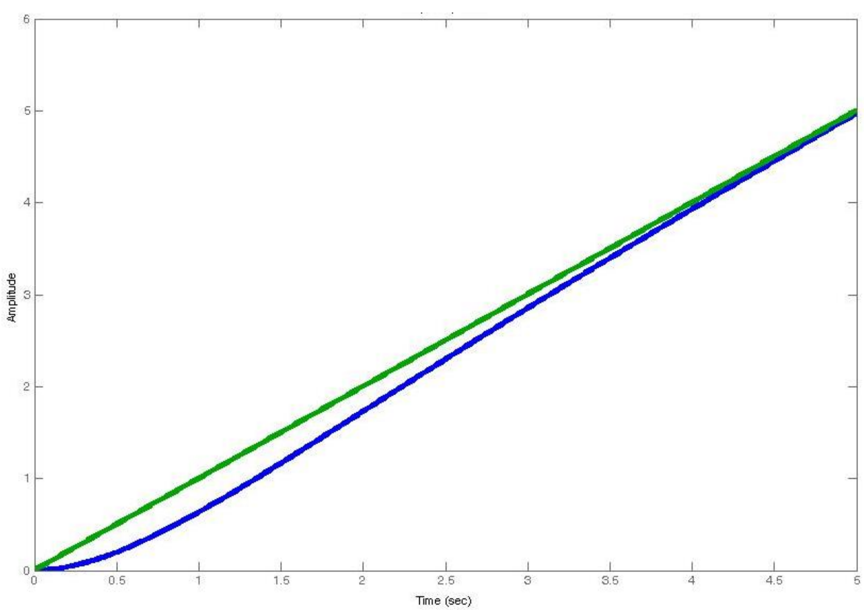
Example: Tracking 2DOF

$$a(s) = (1+s)(1-s)$$
$$b(s) = 2+s$$
$$f(s) = f^{-}(s) = s^2$$
$$m(s) = (2+s)(1+s)^2$$

```
>> a=(1+s)*(1-s), b=2+s, f=s^2,
m=(2+s)*(s+1)^2
>> [p,q]=axbyc(a,b,m,'miny')
p = -2 - s, q =  2 + 2s
>> [t,r]=axbyc(f,b,m,'miny'); r
r =  1 + 2s
>> T=coprime(b*r/(a*p+b*q))
T = 0.3 + 0.6s / 0.3 + 0.6s + 0.3s^2
>> step(tf(T/s),tf(1/s),5)
```

$$a(s)p(s) + b(s)q(s) = m(s)$$
$$f^{-}(s)t(s) + b(s)r(s) = m(s)$$

$$p(s) = -2 - s$$
$$q(s) = 2 + 2s$$
$$r(s) = 1 + 2s$$



Tracking - 1DOF

Design for the same assigned regulator 1DOF, so error control

How to: „selecting from 2DOF controller“

$$p(s) = -2 - s + w(s)(2+s)$$
$$q(s) = 2 + 2s - w(s)(1-s^2)$$
$$r(s) = 1 + 2s - v(s)s^2$$

- From the solutions of the previous tash select such that $q(s) = r(s)$
- Choosing $w(s) = 1 + s^2, v(s) = -s^2$ we get unsatisfactory $p_1(s) = 0$
$$q_1(s) = 1 + 2s + s^2 = r_1(s)$$
- By other choice $u(s) = 1, v(s) = -1$ we obtain
$$p_2(s) = (2+s)s^2, q_2(s) = 1 + 2s + s^4 = r_2(s)$$
- This solution is not proper, but is satisfactory. Proper solution does not exists.
- As expected, the denominator of 1DOF controller has factor s^2

Tracking - 1DOF

- 1DOF controller is directly proposed by a solution of equation

$$a(s)f^{-}(s)x(s)+b(s)q(s)=m(s)$$

- and choosing

$$p(s)=f^{-}(s)x(s)$$

- Because the solution of min. degree

$$x_3(s)=p_3(s)=0$$
$$q_3(s)=1+2s+s^2$$

- is unsatisfactory, we have to find other fom a general solution

$$x(s)=0+w(s)(2+s)$$
$$q(s)=1+2s+s^2+w(s)(1-s^2)s^2$$

- For $w=1$ we obtain

$$x_4(s)=2+s,\ p_4(s)=\left(2+s\right)s^2$$
$$q_4(s)=1+2s+s^4$$

```
>> [x3,q3]=axbyc(a*f,b,m),
p3=x3*f
x3 = 0
q3 = 1 + 2s + s^2
p3 = 0
>> w=1;x4=x3+w*b;
p4=x4*f,q4=q3-w*a*f
p4 = 2s^2 + s^3
q4 = 1 + 2s + s^4
```

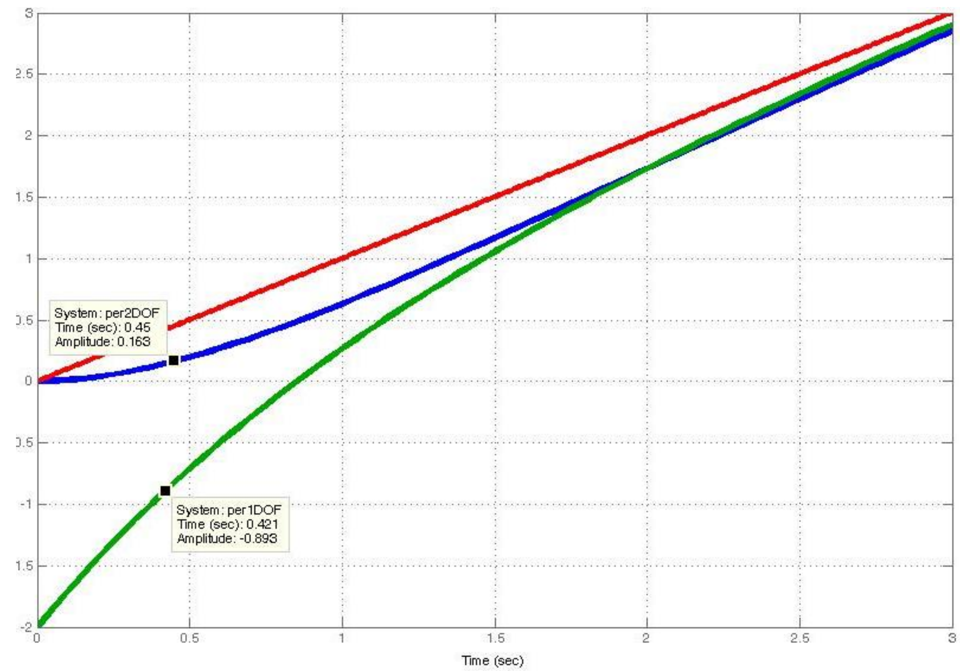
2DOF and 1DOF tracking comparison

- Notice, that
  2DOF controller is proper
  but 1DOF controller is not.
- This is reflected in the CL transfer function in response to ramp signal.

$$\frac{q(s)}{p(s)}=\frac{2+2s}{-2-s},\ \frac{r(s)}{p(s)}=\frac{1+2s}{-2-s}$$

$$\frac{q_4(s)}{p_4(s)}=\frac{1+2s+s^4}{(2+s)s^2}$$

```
>> T2DOF= r*coprime(b/m)
T2DOF =
1 + 2s / 1 + 2s + s^2
>> T1DOF= q4*coprime(b/m)
T1DOF =
1 + 2s + s^4 / 1 + 2s + s^2
>> per2DOF=tf((T2DOF)/s^2);
per1DOF=tf(T1DOF/s^2);
impulse(per2DOF,per1DOF,tf(1/
s^2),3)
```



Physical interpretation and comparison with 1DOF

Conditions for solvability have a nice interpretation:

- Stability $\gcd(a,b)$ means plant stabilizability
- $\gcd(f^{-},b)=1$ is general conditions for the tracking, depends on zero def.: an not pass any (here unstable) mode, which is equal to the zeros
- $f^{-} \mid a$ is also a natural system „driven“ by an input that converges to zero and asymptotically follows just such an unstable signal, which is able to generate by itself. If it is not fulfilled, the system can track only when we give up demand for a stable input

Comparison of two and one degree of freedom

- Special case $\boldsymbol{q} = \boldsymbol{r}$ is easy to differentiate. Equations $\begin{aligned} ap+bq&=m \\ f^{-}t+br&=m \end{aligned}$ have a solution $\boldsymbol{q} = \boldsymbol{r}$ if $ap = f^{-}t$
- Classical structure requires, unstable modes in system or controller transfer function, even if the input can be unstable.

Derivation – System customization

$$y=\frac{b(s)}{a(s)}u \quad + \quad u=-\frac{q(s)}{p(s)}y+\frac{r(s)}{p(s)}u_{new} \quad \Rightarrow \quad y=\frac{b(s)r(s)}{a(s)p(s)+b(s)q(s)}u_{new} \quad = \quad y=\frac{g(s)}{f(s)}u_{new}$$

$$\Rightarrow \quad \frac{b(s)r(s)}{a(s)p(s)+b(s)q(s)}=\frac{g(s)}{f(s)} \quad \Rightarrow \quad \frac{\bar{b}(s)r(s)}{a(s)p(s)+b(s)q(s)}=\frac{\bar{g}(s)}{f(s)} \quad \Rightarrow$$

$$a(s)p(s)+b(s)q(s)=f(s)\bar{b}(s)t(s)$$

$$\Rightarrow \quad \frac{\bar{b}(s)r(s)}{f(s)\bar{b}(s)t(s)}=\frac{\bar{g}(s)}{f(s)} \quad \Rightarrow \quad \frac{r(s)}{t(s)}=\frac{\bar{g}(s)}{1} \quad \Rightarrow \quad r(s)=\bar{g}(s)t(s)$$

Solvable: Always. But if we want stable solutions the conditions must be met.

Stability conditions:

- $f(s)$ stable (of course) and take stable $t(s)$ and $\gcd(a,b)$ has to be stable.
- $\bar{b}(s)$ stable: This means that we can not change the unstable zero!

Example: Model matching

- System $\frac{b(s)}{a(s)}=\frac{(s+1)(s-1)}{(s+2)(s-2)}$ and required t. f. $\frac{g(s)}{f(s)}=\frac{s-1}{(s+2)^2}$
- Relatively prime factors $\frac{b(s)}{g(s)}=\frac{(s+1)(s-1)}{(s-1)}=\frac{(s+1)}{1}=\frac{\bar{b}(s)}{\bar{g}(s)}$
- We choose $t(s)=1$ and solve the equation

$$(s+2)(s-2)p(s)+(s+1)(s-1)q(s)=(s+2)^2(s+1)$$

- Solution $p(s)=-3(s+1), q(s)=2(s+2)$. Feedforward is $r(s)=1$
- The controller is $u=\frac{2(s+2)}{3(s+1)}y+\frac{1}{3(s+1)}u_{new}$
- Verification

$$\frac{(s+1)(s-1)}{-3(s+2)(s-2)(s+1)+2(s+1)(s-1)(s+2)}=\frac{(s+1)(s-1)}{(s+2)^2(s+1)}=\frac{(s-1)}{(s+2)^2}$$