12 – Frequency-domain CTRL design

Martin Hromcik

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Nyquist's stability criterion

Cauchy's theorem of argument, for general function of complex variable ... .

Corollary: Nyquist test (for closed-loop stability assessment, based on open-loop frequency response characteristics). OL = L = b/a, CL = 1/(1+L) = a/(a+b)

- critical point -1 encircled clockwise by the OL Nyquist graph as many times as is
  # CL unstable poles - # CL unstable zeros (=OL unstable poles)
- alternatiuvely: # CL unstable poles = ...
- counterclockwise -> counted with negative sign ↶

Nyquist's stability criterion:

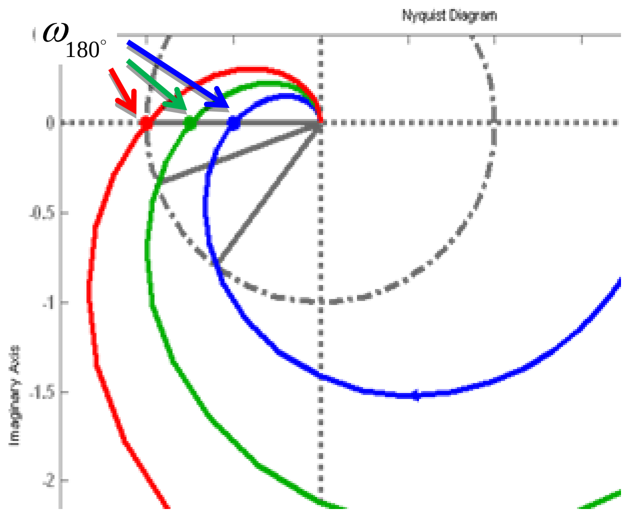
CL stable ⟺ # of encirclements of -1 counterclockwise = # unstable OL poles

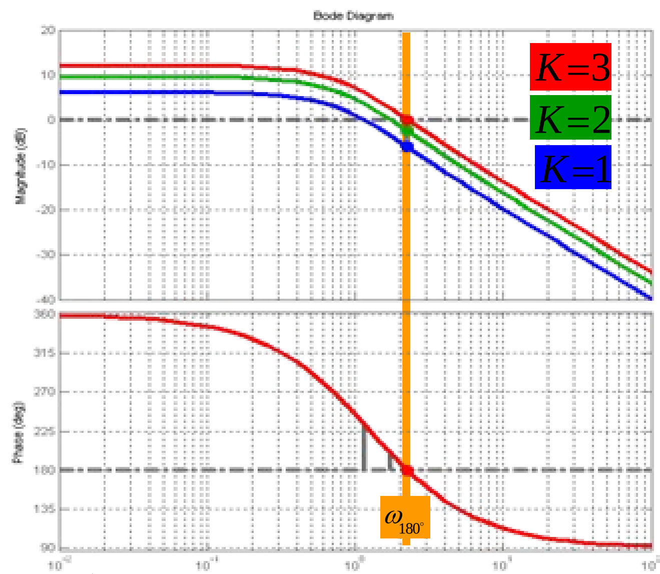
Special case – OL stable:

CL stable ⇔ OL Nyquist does not encircle -1

Gain Margin

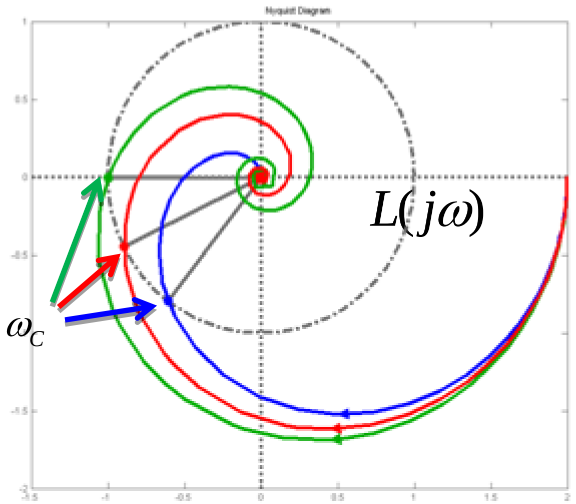
$L(j\omega)$

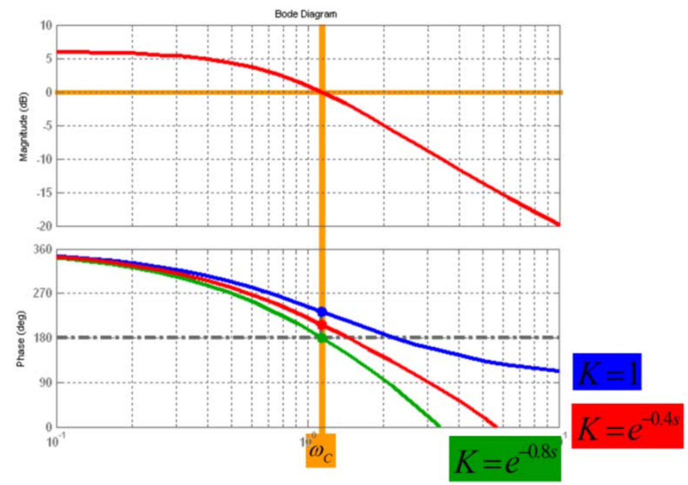




- phase crossover frequency $\omega_{180^\circ} : \angle L(j\omega_{180^\circ}) = -180^\circ$
- Gain Margin: $GM = 1/\left|L(j\omega_{180^\circ})\right|$
- that simple only for OL stable systems ...
- for unstable OL: same idea, more careful treatment (more more crossovers, GM's for each of them, ...)

Phase Margin

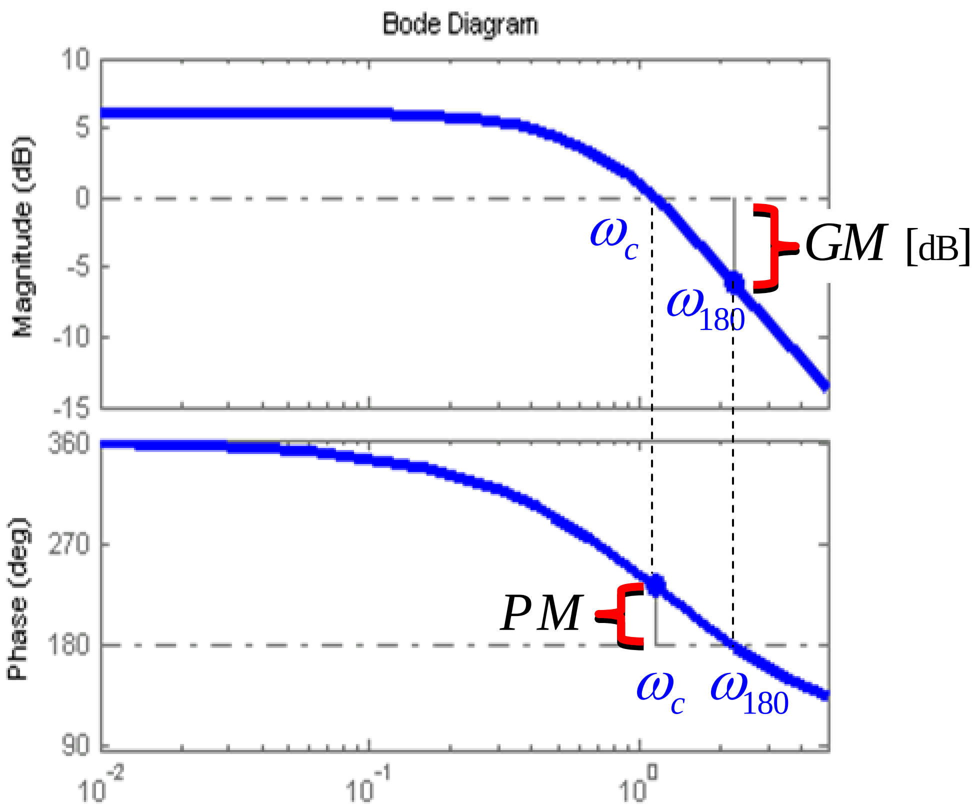


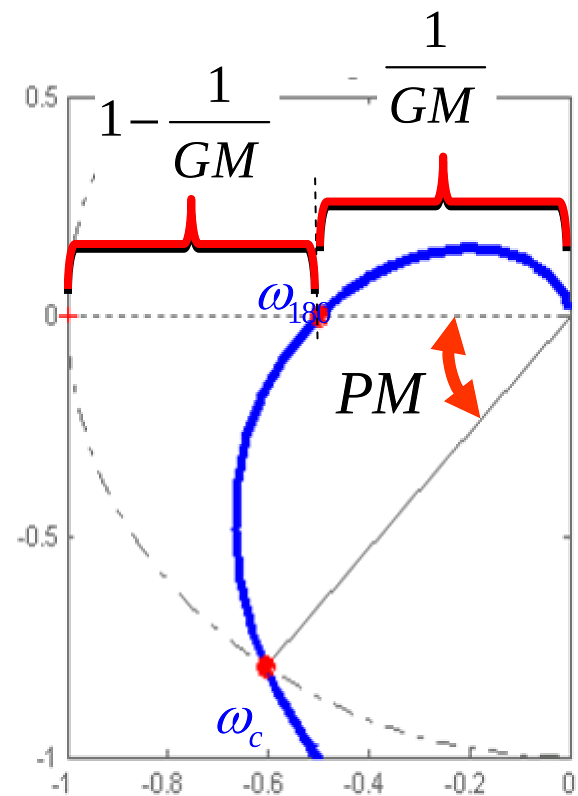


- Gain crossover frequency $\omega_C : |L(j\omega_C)| = 1 = 0\,\mathrm{dB}$
- Phase Margin: $PM = 180^\circ + \angle L(j\omega_C)$
- Time-delay equivalent formulation: $\theta_{\max} = \mathrm{PM}_{\mathrm{rad}} / \omega_c = (\pi/180)\,\mathrm{PM}_{\mathrm{deg}} / \omega_c$

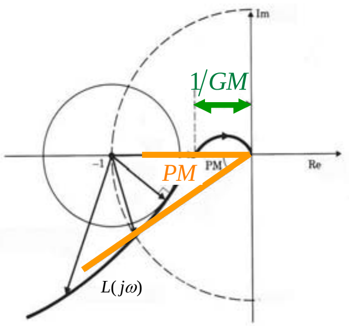
GM and PM

- GM: robustness w.r.t. gain variations; typical requirement GM > 2 (=6dB)
- PM: robustness w.r.t. delays in the loop; typical requirement PM >30°



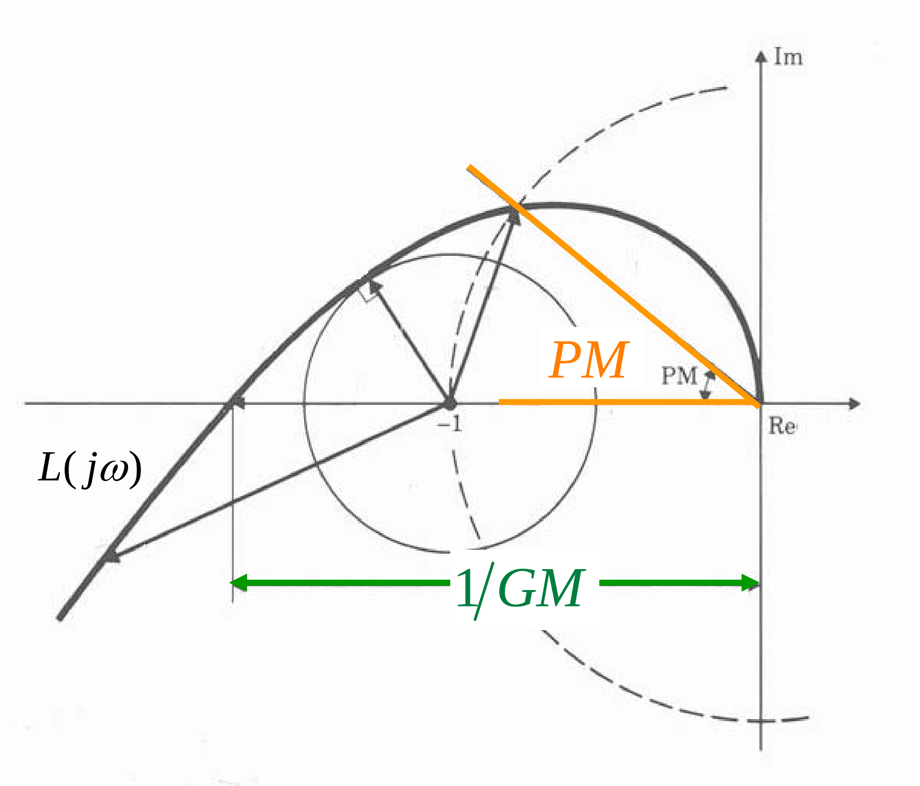


Reversed case ...



<= ... stabilizing stable OL ...

... stabilizing unstable OL ... =>



GM and more crossovers ...

- GM interval $\left(K_{\min}, K_{\max}\right)$
- if CL stable for $L(s) = L_0(s)$
- then CL remains stable for all

$$L(s) = kL_0(s)$$

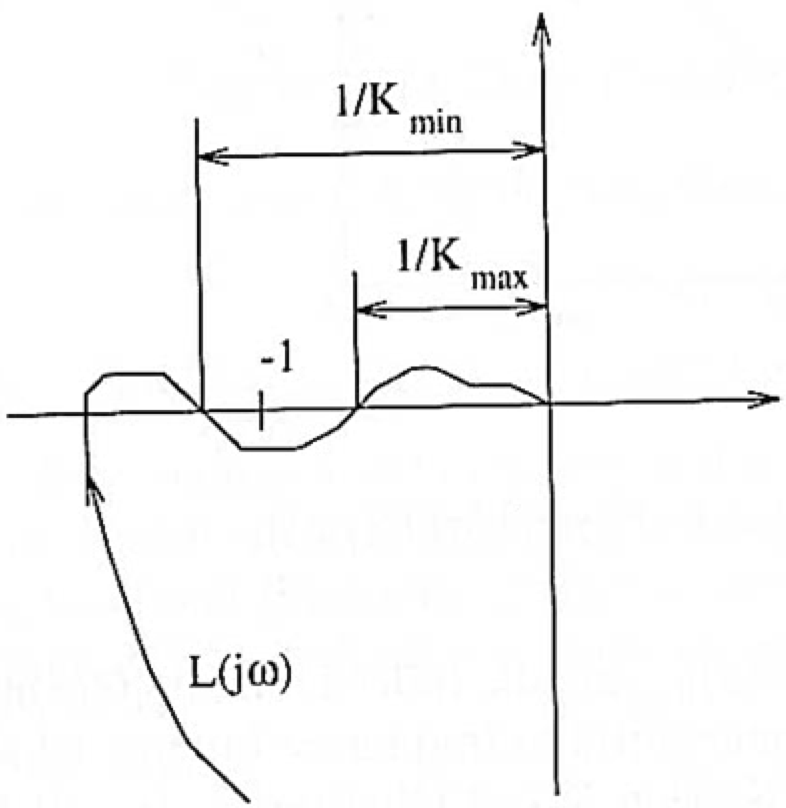
$$k_{\min} < k < k_{\max}$$

$$\begin{pmatrix} 0 \le k_{\min} \le 1 \\ 1 \le k_{\max} \le \infty \end{pmatrix}$$

- CL unstable (already) for

$$L(s) = k_{\min} L_0(s)$$

$$L(s) = k_{\max} L_0(s)$$



PM and more crossovers

- PM interval $(\phi_{\min}, \phi_{\max})$
- If CL stable for $L(s) = L_0(s)$
- then CL remains stable for all

$$L(s) = e^{-j\phi} L_0(s)$$

$$\phi_{\min} < \phi < \phi_{\max}$$

- and becomes unstable for

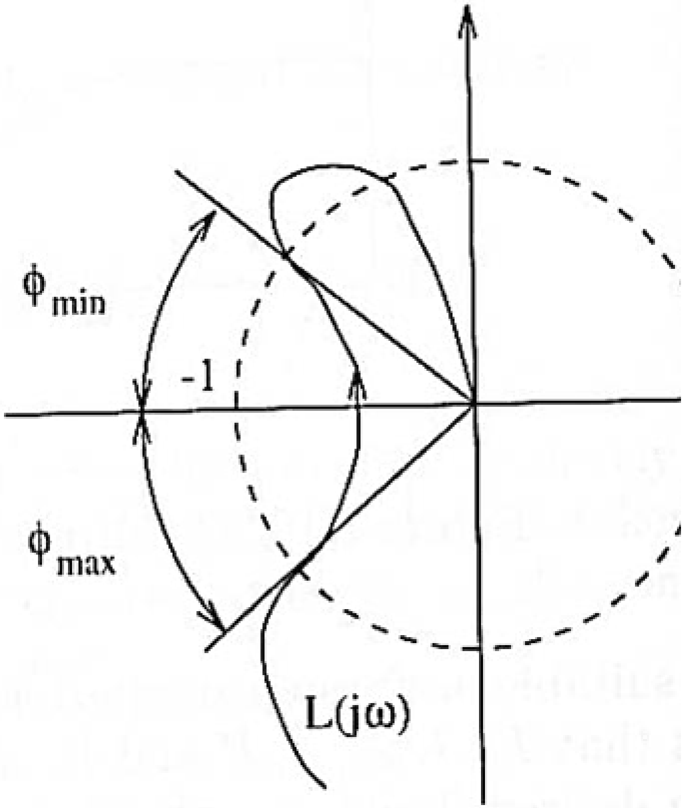
$$L(s) = e^{-j\phi_{\min}} L_0(s)$$

and

$$L(s) = e^{-j\phi_{\max}} L_0(s)$$

- Note that $-\pi \le \phi_{\min} \le 0$

$$0 \le \phi_{\max} \le \pi$$

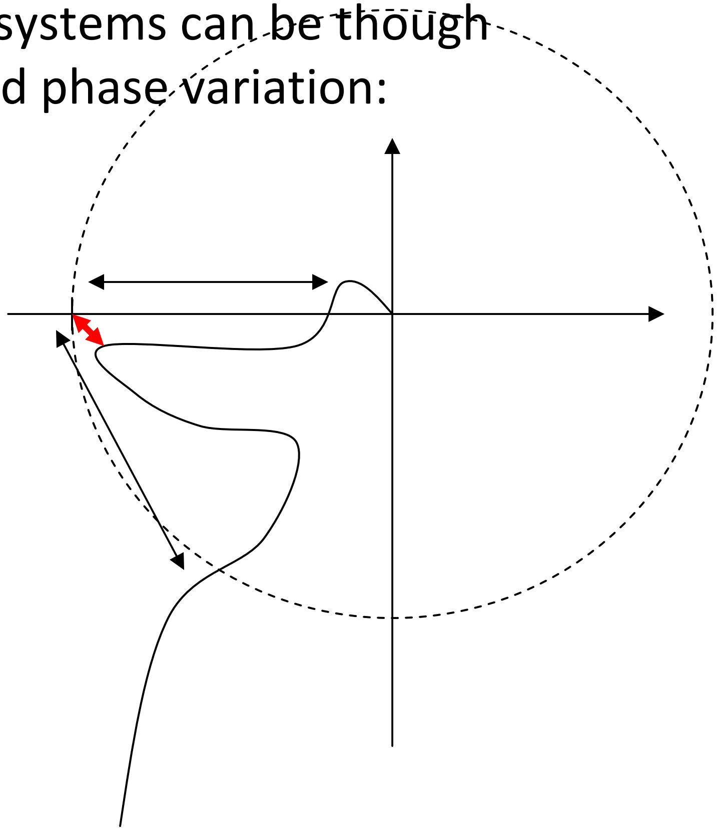


Co je špatného na klasických pojmech ?

- GMs and PMs are most common indicators of controllers robustness in engineering practice.
- Note however that certain GMand PM “safe” systems can be though destabilized easily by combined small gain and phase variation:

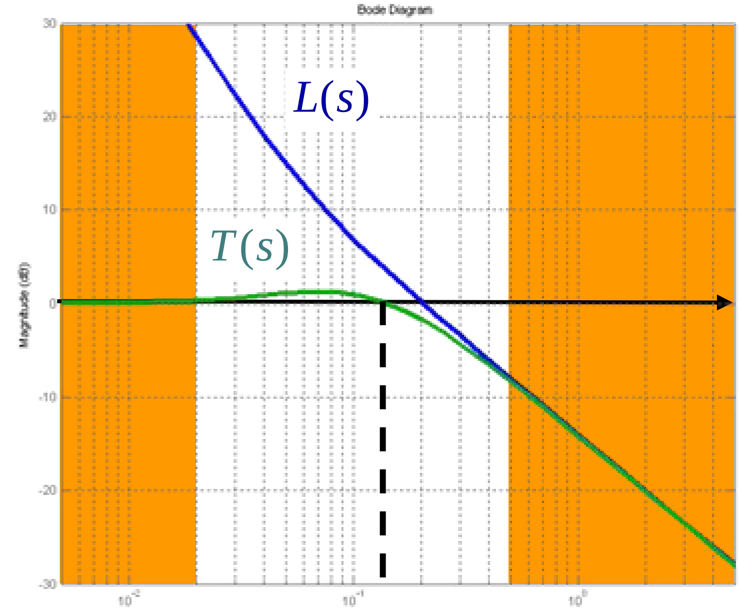
$$L(s) = kL_0(s)e^{-j\phi}$$

$$k \in [k_{\min}, k_{\max}],\ \phi \in [\phi_{\min}, \phi_{\max}]$$



- Solution (generalization of GM/PM concepts): distance (norm) of L from the critical point -1. Hinf robust control ...

CL frequency response



ω_c

OL vs. CL frequency response:

- CL:

$$T(s) = \frac{L(s)}{1+L(s)}$$

- approximately

$$|T(s)| \approx \begin{cases} 1 & \text{for } |L(s)| \text{ large} \\ |L(s)| & \text{for } |L(s)| \text{ small} \end{cases}$$

- typically

$$|L(s)| \begin{cases} \text{large for high freqs} & \omega \ll \omega_c \\ \text{small for low freqs} & \omega \gg \omega_c \end{cases}$$

- therefore

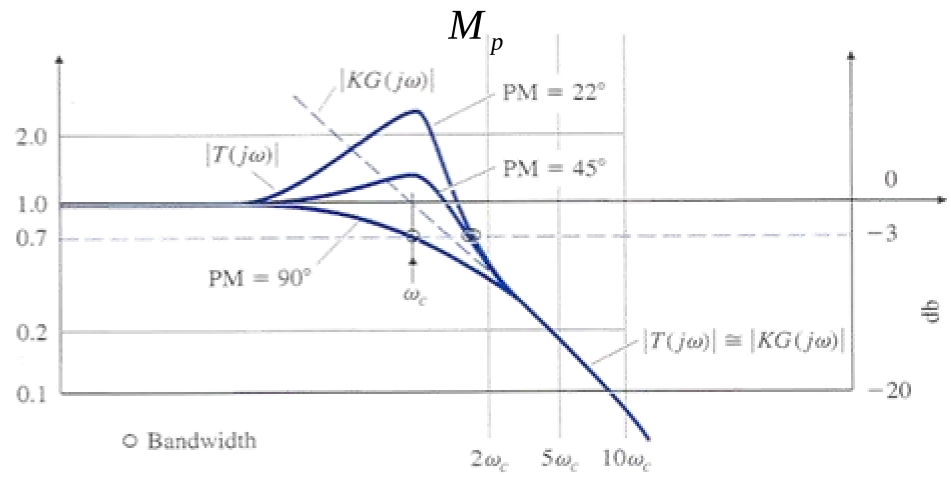
$$|T(s)| \approx \begin{cases} 1 & \text{for low freqs} & \omega \ll \omega_c \\ |L(s)| & \text{for high freqs} & \omega \gg \omega_c \end{cases}$$

Crossover frequency behavior

- around ωc , $|L(s)| \approx 1$, $|T(s)|$ depends on PM
- PM 90° ➡ $\angle L(j\omega_c) = -90^\circ$ ➡ $|T(j\omega_c)| = 1/\sqrt{2} \approx 0.707$
- PM 45° ➡ $\angle L(j\omega_c) = -135^\circ$ ➡ $|T(j\omega_c)| \approx 1.31$
- for 2nd order system:

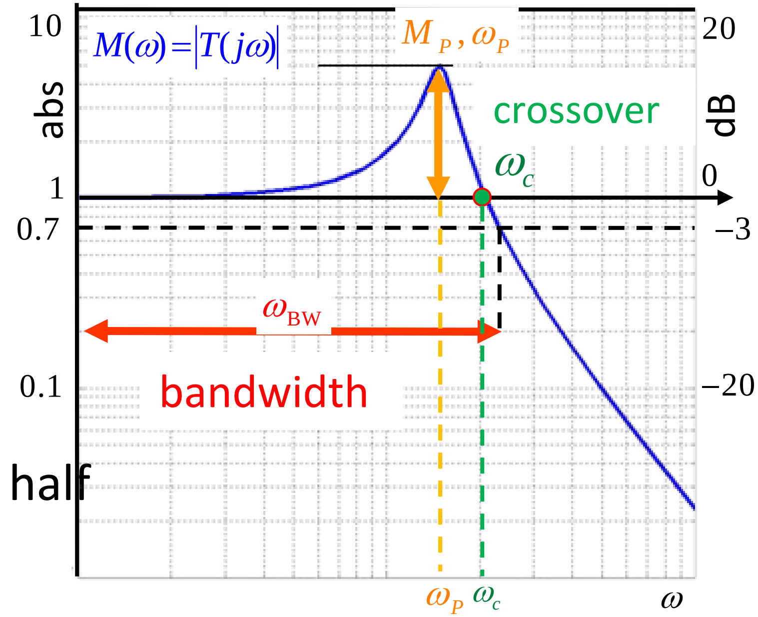
$$PM = \arctan \frac{2\varsigma}{\sqrt{-2\varsigma^2 + \sqrt{1+4\varsigma^4}}}$$

$$M_p = \frac{1}{2\varsigma\sqrt{1-\varsigma^2}}$$



Bandwidth and crossover frequency

- bandwidth = those frequencies (of harmonic signals) wich the CL is capable to track reasonably ...
- typically, CL = low pass filter: good tracking for ω small $(|T| \approx 1)$, but not for large $(|T| \approx 1)$
- formal definition of classical control: bandwidth = frequency whee the output has half energy of input, $y^2 = 0.5u^2$



- Hence

$$|Y(j\omega)| = \sqrt{1/2}\,|U(j\omega)| = 0.707\,|U(j\omega)| \iff |Y(j\omega)|_{\mathrm{dB}} = |U(j\omega)|_{\mathrm{dB}} - 3\,\mathrm{dB}$$

- bandwidth defines “responsivenes” of CL, speed ...
- higher bandwidth – more aggressive control law – implies faster transients, but often higher sensitivity to parameter changes and noises. lower bandwidth typically opposite.
- Crossover frequency: $\omega_c : T(j\omega_c) = 1\ (0\,\mathrm{dB})$
- Resonance peak: $M_P = \max\{|T(j\omega)|\}, \omega_P : M_P = |T(j\omega_P)|$