Exercises for lectures
13 – Design using frequency methods

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Setting of the closed loop bandwidth

- At the transition frequency in the open loop is (from definition) \(|L(j\omega_c)| = 1\)
- Value is \(|T(j\omega_c)| = \frac{|L(j\omega_c)|}{|1 + L(j\omega_c)|}\)
  
  but it also depends on the phase \(\angle L(j\omega_c)\), thus on PM (Phase Margin)
- For PM = 90° is \(L(j\omega_c) = -j\) and small phase - 90°, so
  \[|T(j\omega_c)| = \frac{|-j|}{|1 - j|} = \frac{1}{\sqrt{2}} \approx 0.707\]

- In this case the closed loop bandwidth equals to the transition frequency of open loop exactly!
  \(\omega_{BW} = \omega_c\)

- For small PM, the value \(|T(j\omega_c)|\) rises and appears a resonance peak. This bandwidth \(\omega_{BW}\) moves to the right, but usually does not exceed \(2\omega_c\).
- It is usually \(\omega_c \leq \omega_{BW} \leq 2\omega_c\).
- Therefore, we set \(\omega_c\) (OL !!!) in order to ensure the required \(\omega_{BW}\) (CL!!!)
Relationship of $\omega_c$ and $\omega_{BW}$

- Bode plot $|T(j\omega)|$
- with marked $\omega_c$
- and values $\omega_{BW}$
- for various PM
- It is usually $\omega_c \leq \omega_{BW} \leq 2\omega_c$
- For a 2nd order system without zeros, the dependency on $\zeta$ is shown in the figure.

$$\frac{\omega_c}{\omega_{BW}} = \sqrt{\frac{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2} + 2}} \in \left(\frac{1}{2}, 1\right)$$

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Repetition: steady behavior of the Bode plot

- Initial slope is 0 so the system is of type 0 "initial value" (without a pole at 0) asymptote is 15 dB and thus:
  \[ K_p = 15 \text{ dB} = 10^{15/20} = 5.623 \]
- steady-state error to a step is:
  \[ e_{\text{step,ss}} = \frac{1}{1 + K_p} = 0.151 \]

- Initial slope is 20 dB/dek and so the system is of type 1 (with one pole at 0)
- stretched "initial asymptote" intersects the zero line for frequency \( \omega = 10 \) and thus:
  \[ K_v = 10 \]
- Steady-state error to a ramp:
  \[ e_{\text{ramp}}(\infty) = \frac{1}{K_v} = 0.1 \]
Comparison of time and frequency responses

Automatické řízení - Kybernetika a robotika

Prototype second-order system

\[ y(t) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \]

As \( \omega_n \) gets larger, pole distance from origin gets larger.

As \( \xi \) gets larger, angular distance from negative real axis gets smaller.

Frequency response

\[ BW = \omega_n \left[ (1 - 2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2} \right]^{1/2} \]

As \( \omega_n \) gets larger, BW gets larger.

As \( \xi \) gets larger, BW gets smaller.

Bandwidth and rise time are inversely proportional.

Therefore, the larger the bandwidth is, the faster the system will respond.

Increasing \( \omega_n \) increases BW and decreases \( t_r \).

Increasing \( \xi \) decreases BW and increases \( t_r \).
Example: Setting $K_p$ by a P regulator

- System is
  \[ G(s) = \frac{5}{s + 2} \]
  
  \[ K_p = 2.5, \quad K_p,\text{dB} = 20\log 2.5 = 8\text{dB} \]
  
  \[ e_{ss} = \frac{1}{1 + K_p} = 0.29 \]

- We want
  \[ e_{ss,2} = 0.01 \rightarrow K_{p,2} = \frac{1 - e_{ss,2}}{e_{ss,2}} = 99 \]
  
  \[ K_{p,2,\text{dB}} = 20\log 99 = 40\text{dB} \]

- We use
  \[ K = \frac{K_{p,2}}{K_p} = \frac{99}{2.5} = 39.6 \]
  
  \[ K_{\text{dB}} = K_{p,2} - K_{p,\text{dB}} = 40 - 8 = 32\text{dB} \]

(Beware - the result is very fast, with a large action peak)
Example: Setting $K_v$ by a P regulator

- System $G(s) = \frac{58390}{s(s+36)(s+100)}$ has $K_v = 16.22 \rightarrow e_{\text{ramp}}(\infty) = \frac{1}{K_v} = 0.0617$

- If we want to reduce the steady-state error to ramp 10x, we must set $K_v = 162.2$

- We increase the gain 10x, what leads to $L(s) = \frac{583900}{s(s+36)(s+100)}$

- We obtain $K_v = 162.2$

- but beware, the result is unstable! Here, P controller will not solve the task!
For the position control system in the figure set the preamp gain so that the resulting system reaches 9.5% overshoot by a step of reference.

From the required overshoot we calculate the damping (of dominant poles)

\[ \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = \frac{-\ln(0.095)}{\sqrt{\pi^2 + \ln^2(0.095)}} = 0.5996 \approx 0.6 \]

and from that we obtain \( PM \)

\[ PM = \arctan \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}} = \arctan \frac{2 \times 0.6}{\sqrt{-2 \times (0.6)^2 + \sqrt{1 + 4 \times (0.6)^4}}} = 1.0326 \approx 59.2^\circ \]

The open loop transfer function has an indefinite \( K \).

To draw the Bode plot and perform a design on the plot we have to choose some \( K \).

Let choose \( K = 3.6 \) and obtain

\[ L(s) = \frac{100K}{s(s + 36)(s + 100)} \]

\[ L_{K=3.6}(s) = \frac{360}{s(s + 36)(s + 100)} \]
Example: Setting of gain for required PM

- We draw a Bode plot and find a frequency, for which
  \[ L_{K=3.6}(s) = \frac{360}{(s(s+36)(s+100))} \]
  \[ \angle L(j\omega') = -180^\circ + 59.2^\circ = -120.8^\circ \]
- From the graph we subtract \( \omega' = 14.8 \text{ rad/s} \)
- For this frequency the amplitude is \( |L(\omega')| = M(\omega') = 0.0062 = -44.2 \text{ dB} \) and therefore we must increase gain by 44.2 dB, so cca 162.2x.
- Then we obtain
  \[ L(s) = \frac{58390}{s(s+36)(s+100)} \]
- Simulation verifying the correct design is necessary.
- We continue later with this example and for this purpose we measure
  \[ K_v = 16.22 \rightarrow e_{\text{ramp}}(\infty) = 0.0617 \]
Example: Setting PD

- System transfer function (aircraft attitude)
  \[ G(s) = \frac{4500}{s(s+361.2)} \]

- Requirements
  \[ e_{ramp,ss} \leq 0.000443 \rightarrow K_v \geq 1/e_{ramp,ss} = 2257 \]
  \[ PM \geq 80 \]

- First set \( K_p = 181.19 \),
  to increase \( K_{v,1} = 12.5 \) to \( K_v = 2258 \) and to ensure
  the required regulation error.

- Then search for the part
  \[ (1 + K_ds) \]

- Obtained PD regulator for the system is
  \[ K_pG(s) = \frac{815350}{s(s+361.2)} \]
We draw the Bode plot for system

\[ L(s) = K_p \left(1 + K_D s\right) G(s) = \frac{815350}{s(s + 361.2)}(1 + K_D s) \]

for \( K_D = 0 \).

We find \( \omega_D \), at which

\[ PM = \text{required} - (\text{regulator phase at } \omega_D) = 80^\circ - 45^\circ = 35^\circ \text{ where} \]

the phase is \(-180^\circ + 35^\circ = -145^\circ\).

It is \( \omega_D = 516 \).

We calculate

\[ K_D = \frac{1}{\omega_D} = \frac{1}{516} = 0.0019 \]

Resulting \( L \) has a Bode plot.

The requirement is fulfilled: \( PM = 84.9^\circ \).
For a transfer function

\[ G(s) = \frac{1.5 \times 10^7}{s(s^2 + 3408.3s + 1204000)} \]

Consider, we already designed \( K_P = 1 \) and now we set \( K_D \) in PD regulator \((1 + K_D s)\) for good PM

- We draw Bode plot for following values \( K_D = 0, 0.002, 0.005, 0.02 \)
- Uncompensated system \((K_d = 0)\) has \( PM = 7.78^\circ \)
- To reach \( PM = 80^\circ \), regulator should add \( 72.22^\circ \) to the new \( \omega_c \)
- From figure it follows, that it is impossible. High regulator gain shifts \( \omega_c \) to higher frequencies,
- where phase of the uncompensated system declining faster than it is increased by the compensator.
Example: Setting PI

- For a transfer function \( G(s) = \frac{815350}{s(s + 361.2)} \)

- Find a PI regulator, that increases \( PM = 22.6^\circ \) to \( PM_{\text{new}} = 65^\circ \)

  Draw a Bode graph

\[
L(s) = \frac{815350K_p(s + K_I/K_p)}{s^2(s + 361.2)}
\]

- First for \( K_p = 1 \) and \( K_I = 0 \)

- From requirement \( PM_{\text{new}} = 65^\circ \) find \( \omega_{c,\text{new}} = 170 \) rad/s and calculate

\[
K_p = 10^{-\frac{|G(j\omega_{c,\text{new}})|_{\text{dB}}}{20}} = 10^{-\frac{21.5}{20}} = 0.084
\]

- \( K_I \) choose so that the corner freq. is less than a decade \( \omega_{c,\text{new}} \)

\[
K_I/K_p = \frac{\omega_{c,\text{new}}}{10}
\]

\[
K_I = K_p \frac{\omega_{c,\text{new}}}{10} = 0.084 \times 170/10 \approx 1.42
\]
Example: Setting PI

- For this $K_I = 1.42$ calculate the transfer function and draw the Bode plot

$$L(s) = \frac{815350K_p(s + K_I/K_p)}{s^2(s + 361.2)} = \frac{68489(s + 16.9)}{s^2(s + 361.2)}$$

- We obtain $PM_{\text{new}} = 59$, what does not satisfy the requirement.
- Let's try to use a smaller $K_I$ (= move the corner frequency to left).
- For example $K_I = 0.07$ leads to the transfer function

$$L_2(s) = \frac{68489(s + 0.833)}{s^2(s + 361.2)}$$

- with $PM_{\text{new}} = 64.3$
• See the attached document.
Example: Lag regulator design

Task: For a plat give by a transfer function

\[ F(s) = \frac{1}{s(s + 2)(s + 30)} \]

design a Lag regulator satisfying these requirements: \( e_{ss,ramp} \leq 0.05, \ P_M \geq 45^\circ \)

Solution:

1. Find the value of the gain providing the desired deviation:

\[ L_1(s) = KF(s) = \frac{K}{s(s + 2)(s + 30)} \]

\[ e_{ss,ramp} = \frac{1}{K_v} = \lim_{{s \to 0}} sL_1(s) = \frac{1}{K} = \frac{60}{K} \leq 0.05 \Rightarrow K \geq \frac{60}{0.05} = 1200 \]

This OL transfer function gives incorrect PM and GM

\[
\begin{align*}
>> & \quad K=1200; F=1/s/(s+2)/(s+30); L1=K*F \\
& \quad L1 = 1200 / 60s + 32s^2 + s^3 \\
>> & \quad [GM, PM, om_cp, om_cg]=margin(tf(L1)) \\
& \quad GM = 1.6000 \\
& \quad PM = 6.6449 \\
& \quad om_cp = 7.7460 \\
& \quad om_cg = 6.1031 \\
>> & \quad GM_{dB} = 20*\log10(GM) \\
& \quad GM_{dB} = 4.0824
\end{align*}
\]
2. Draw a Bode plot $L$

$$L_1(s) = \frac{1200}{s(s+2)(s+30)}$$

- From the required PM we calculate necessary phase and we find new $\omega_{c,new} = 1.28 \text{ rad/s}$
- At this frequency, we find the necessary attenuation

$$|\Delta C(j\omega_{c,new})|_{dB} = -22.1 \text{ dB}$$

3. We calculate the parameter $a$ from the measurements or from a transfer fcn.

$$a = |\Delta C(j\omega_{c,new})| = 10^{\frac{1}{20}}|\Delta C(j\omega_{c,new})|_{dB} = 10^{-\frac{22.1}{20}} = 0.0785$$

$$>> \text{aa}=1/\text{abs(value(L1,j*1.28))}$$

$$\text{aa} = 0.0761$$
4. We calculate zero

\[ z_c = \frac{\omega_{c,new}}{10} = 0.128 \]

and pole

\[ p_c = az_c = 0.0785 \times 0.128 = 0.0101 \]

5. The final regulator is

\[ C_{lag}(s) = \frac{as + p_c}{s + p_c} = \frac{0.0785s + 0.0101}{s + 0.0101} \]

6. Finally, we verify if the regulator satisfies the requirements.
Example: Lag regulator design

\[ \text{Kv} = 20.0000 \], \[ \text{e}_{\text{ss}} \text{-ramp} = 0.0500 \]

Bode Diagram

Frequency (rad/s)

Magnitude (dB)

Phase (deg)

Phase Margin (deg): 49
Delay Margin (sec): 0.65
At frequency (rad/s): 1.32
Closed loop stable? Yes

\[ \text{Kv} = \text{value(coprime(s*L2),0), e}_{ss \text{-ramp}} = 1/\text{Kv} \]
In the positioning control system, the gain was by previous method adjusted so that the system has overshoot 9.5% and

\[ L(s) = \frac{58390}{s(s + 36)(s + 100)} \]

\[ K_v = 16.22 \rightarrow e_{\text{ramp}}(\infty) = \frac{1}{K_v} = 0.0617 \]

Add Lag compensation so that the steady state value to the ramp is 10x smaller and the overshoot does not increase. The steady state leads to \( K_v = 162.2 \), so we have to increase the gain 10x and then we obtain

\[ L(s) = \frac{583900}{s(s + 36)(s + 100)} \]

The overshoot requirement 9.5% leads to

\[ \zeta = 0.6 \rightarrow PM = 59.2^\circ \]

Because Lag decreases PM only little, but still (we expect a deterioration \( \Delta PM = -5^\circ \leftrightarrow -12^\circ \) ), we consider rather \( PM = 59.2^\circ + 10^\circ = 69.2^\circ \)

Let's find a frequency \( \omega' \), for which the phase is

\[ \angle L(j\omega') = -180^\circ + 69.2^\circ = -110.8^\circ \]
• From required phase $-110.8^\circ$
  
• we determine frequency $\omega' = 9.8 \text{rad/s}$

and then the value

$$20 \log M(\omega') = 24 \text{dB}$$

• From the definition, PM for $\omega'$ should be $20 \log M(\omega') = 0 \text{dB}$

• Lag should have at the frequency $\omega' = 9.8 \text{rad/s}$ attenuation $-24 \text{dB}$
Lag compensation

- Draw the asymptote for higher frequencies in
  \[ 20\log M(\omega') = -24\text{dB} \]

- The upper corner frequency \( \omega' \) is chosen by a decade left from \( \omega' = 9.8\text{rad/s} \), it is \( 1/T = 0.98\text{rad/s} \)
- From there we continue with the slope \(-20\text{dB/dec}\) to 0dB, what we reach for \( 1/\alpha T = 0.062\text{rad/s} \)
- After substitution we obtain
  \[
  C(s) = \frac{s + 1/T}{s + 1/\alpha T} = \frac{s + 0.98}{s + 0.062}
  \]

- It has correct shape, but not the gain, so we set up the DC gain of the compensator
  \[
  K_C = 1/\alpha = p/z \rightarrow D_C(0) = 1 = 0\text{dB}
  \]
  \[
  D_C(s) = K_C C(s) = \frac{0.063(s + 0.98)}{s + 0.062}
  \]
The result is:

\[
\frac{583900}{s(s+36)(s+100)} \times \frac{0.063(s+0.98)}{s+0.062} = \frac{36787(s+0.98)}{s(s+36)(s+100)(s+0.062)}
\]
• Lets get back to the positioning control system and design a regulator according to specifications:
  - OS 20%, $K_v = 40$, $T_p = 0.1s$
  - First set up gain so, that $K_v = 40$
    $$K_v = \lim_{s \to 0} |sL(s)| = 0.0278K = 40 \rightarrow K = 1440$$
  - Lets substitute it and continue
    $$L(s) = \frac{100K}{s(s + 36)(s + 100)}$$
  - From the given specification we calculate $PM$ a $\omega_{BW}$:
    $$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} \approx 0.456 \rightarrow PM = \arctan \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}} \approx 48.1^\circ$$
    $$\omega_{BW} = \frac{\pi}{T_p \sqrt{1 - \zeta^2}} \sqrt{\left(1 - 2\zeta^2\right) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 46.6 \text{rad/s}$$
• Draw a Bode plot for

• This uncompensated system has $PM = 34.1^\circ$

• By Lead compensation we increase PM to required value

• Since Lead also increases $\omega_C$, we add also some compensation factor.

• To compensate the smaller phase for larger frequencies $\omega_C$ we choose the factor as $10^\circ$.

• We require the regulator phase increase of $48.1^\circ - 34^\circ + 10^\circ = 24.1^\circ$
Lead compensation

We require the regulator phase increase of $48,1^° - 34^° + 10^° = 24,1^°$.

Generally the compensated system should have $PM = 48.1$ and $\omega_{BW} = 46.6$ rad/s.

It should not produce satisfactory results, we have to repeat the design with other correction factor.

From the phase growth requirement we have $\phi_{\text{max}} = 24.1^°$ and from it

$$\beta = \frac{1 - \sin \phi_{\text{max}}}{1 + \sin \phi_{\text{max}}} = 0.42$$

It follows that

$$\left| D(\omega_{\text{max}}) \right| = \frac{1}{\sqrt{\beta}} = 3.76 \text{ dB}$$

If we choose $\omega_{C,\text{new}} = \omega_{\text{max}}$, then at this frequency the amplitude of the uncompensated system should be -3,76 dB.

According to that we find $\omega_{\text{max}}$.
Lead compensation

- On the Bode plot
  \[ L(s) = \frac{144000}{s(s + 36)(s + 100)} \]
  - We measure \( \omega_{\text{max}} = 39 \text{ rad/s} \)
  - Then from \( \omega_{\text{max}} \) and \( \beta = 0.42 \) we calculate
    \[ \omega_{\text{max}} = \frac{1}{T \sqrt{\beta}} \]
    \[ \frac{1}{T} = 25.3, \quad \frac{1}{T \beta} = 60.2 \]
  - and in the end, we obtain the search factor
    \[ D(s) = \frac{1}{\beta} \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = 2.38 \frac{s + 25.3}{s + 60.2} \]
The result is:

- Simulation:

$$OS\% = 22.6, PM = 45.5^\circ, \omega_C = 39 \text{ rad/s}$$

$$\omega_{BW} = 68.8 \text{ rad/s}, T_p = 0.075 \text{ s}, K_v = 40$$