15 – State feedback

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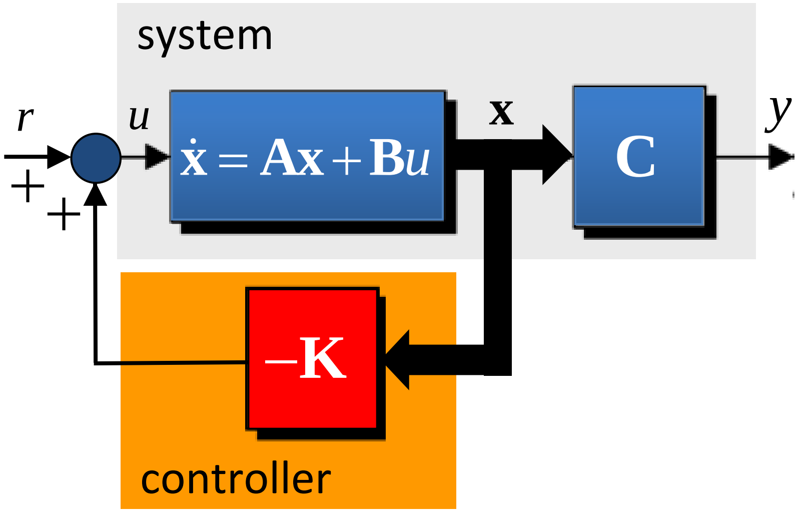
State feedback

$$u = -\mathbf{K}\mathbf{x} + r = -\begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + r$$



- static FB
- "full-information" fedback
- fundamental goal: dynamics modification (artificial damping, pole-placement)

State feedback

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad u = -\mathbf{K}\mathbf{x} + r$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \mathbf{A}\mathbf{x} + \mathbf{B}\left(-\mathbf{K}\mathbf{x} + r\right) = \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x} + \mathbf{B}r$$

$$\boxed{\dot{\mathbf{x}} = \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x} + \mathbf{B}r}$$

$$\dot{\mathbf{x}} = \boxed{\mathbf{A}}\mathbf{x} + \mathbf{B}u \quad \Rightarrow \quad \dot{\mathbf{x}} = \boxed{\left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)}\mathbf{x} + \mathbf{B}r$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

$$\mathbf{A}_{\mathbf{new}} = \mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix}$$

Pole-placement: system in CTRL canonical form

$$\det\left(s\mathbf{I}-\mathbf{A}_{\mathbf{new}}\right)=\det\left(s\mathbf{I}-\left(\mathbf{A}-\mathbf{BK}\right)\right)$$

$$\mathbf{A}=\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\det\left(\mathbf{sI}-\mathbf{A}\right)=s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_n$$

Pole-placement: system in CTRL canonical form

$$\mathbf{A}_{\mathbf{new}} = \mathbf{A} - \mathbf{B}\mathbf{K} =$$

$$= \left[\begin{array}{cccc|c} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right] - \left[\begin{array}{c} 1 \\ \hline 0 \\ \vdots \\ 0 \\ 0 \end{array}\right] \left[\begin{array}{ccc} k_1 & \dots & k_n \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} -(a_{n-1}+k_1) & -(a_{n-2}+k_2) & \cdots & -(a_1+k_{n-1}) & -(a_0+k_n) \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

- overall characterisitic polynomial:

$$p_{new}(s) = \det\left(\mathbf{sI} - \mathbf{A}_{\mathbf{new}}\right) = s^n + \left(a_{n-1} + k_1\right) s^{n-1} + \cdots + \left(a_1 + k_{n-1}\right) s + \left(a_0 + k_n\right)$$

- select K so that

$$p(s) = s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + p_0$$

$$k_1 = p_{n-1} - a_{n-1}$$

$$\vdots$$

$$k_{n-1} = p_1 - a_1$$

$$k_n = p_0 - a_0$$

Pole-placement: system in general form

General case: transformation into CTRL canonical form.

$$\mathbf{x} = \mathbf{T}\mathbf{x}_{\text{con}}$$
$$\mathbf{A} = \mathbf{T}\mathbf{A}_{\text{con}}\mathbf{T}^{-1}$$
$$\mathbf{B} = \mathbf{T}\mathbf{B}_{\text{con}}$$

$$T = \begin{bmatrix} B & AB & \cdots & A^{N-1}B \end{bmatrix} \times \\ \times \begin{bmatrix} B_{con} & A_{con}B_{con} & \cdots & A_{con}{}^{N-1}B_{con} \end{bmatrix}^{-1}$$

$$\mathbf{A}_{\text{con}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \qquad \mathbf{B}_{\text{con}} = \mathbf{T}^{-1}\mathbf{B}$$

$$\mathbf{K}_{\text{con}}$$

$$\boxed{\mathbf{K} = \mathbf{K}_{\text{con}}\mathbf{T}^{-1}}$$

$$\dot{\mathbf{x}}_{\text{con}} = \left(\mathbf{A}_{\text{con}} - \mathbf{B}_{\text{con}}\mathbf{K}_{\text{con}}\right)\mathbf{x}_{\text{con}} + \mathbf{B}_{\text{con}}r \quad \Rightarrow$$

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{T}\left(\mathbf{A}_{\text{con}} - \mathbf{B}_{\text{con}}\mathbf{K}_{\text{con}}\right)\mathbf{T}^{-1}\mathbf{x} + \mathbf{T}\mathbf{B}_{\text{con}}r \\ &= \left(\mathbf{A} - \mathbf{B}\mathbf{K}_{\text{con}}\mathbf{T}^{-1}\right)\mathbf{x} + \mathbf{B}r \\ &= \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x} + \mathbf{B}r \end{aligned}$$

Ackermann formula

$$p_{CL}(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0$$

$$\Downarrow$$

$$p_{CL}(\mathbf{A}) = \mathbf{A}^n + p_{n-1}\mathbf{A}^{n-1} + \cdots + p_1\mathbf{A} + p_0\mathbf{I}_n$$

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

$$\Downarrow$$

$$\boxed{\mathbf{K} = [0 \quad \ldots \quad 0 \quad 1]\,\mathcal{C}^{-1} p_{CL}(\mathbf{A})}$$

Zeros

$$\det\begin{bmatrix} s\mathbf{I}-\mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} = 0$$

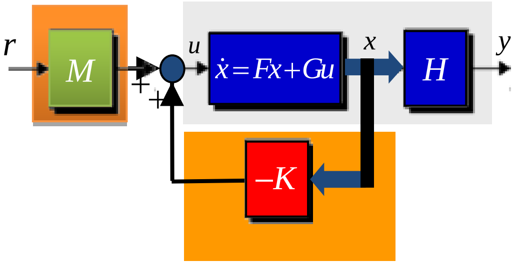
$$\det\begin{bmatrix} s\mathbf{I}-(\mathbf{A}-\mathbf{B}\mathbf{K}) & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} = 0$$

$$\det\begin{bmatrix} s\mathbf{I}-\mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} = \det\begin{bmatrix} s\mathbf{I}-\mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{K} & \mathbf{I} \end{bmatrix} = \det\begin{bmatrix} s\mathbf{I}-\mathbf{A}+\mathbf{B}\mathbf{K} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} = \det\begin{bmatrix} s\mathbf{I}-(\mathbf{A}-\mathbf{B}\mathbf{K}) & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

State feedback and non-robust reference tracking

$$P(0)=\left(\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+D\right)\Big|_{s=0}=-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}+\mathbf{D}$$

$$T(0)=\left(\mathbf{C}(s\mathbf{I}-(\mathbf{A}-\mathbf{B}\mathbf{K}))^{-1}\mathbf{B}+D\right)\Big|_{s=0}=-\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{K})^{-1}\mathbf{B}+\mathbf{D}$$

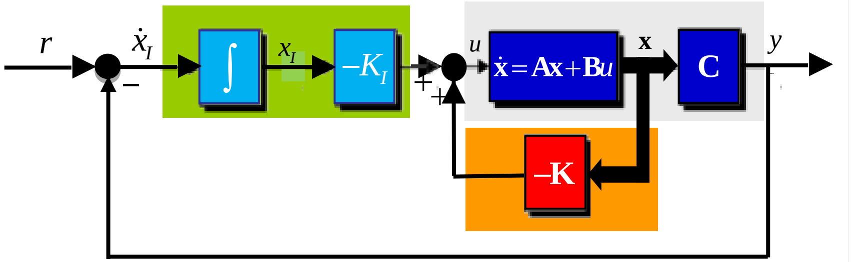


$$M=1/T(0)$$

$$T_{celk}(0)=M(0)T(0)=T(0)/T(0)=1$$

State feedback and robust reference tracking (integral control)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$
$$\dot{x}_I = -\mathbf{C}\mathbf{x} + r$$



$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_I \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_I \end{bmatrix}$$