Exercises for lectures
15 – State feedback

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Example: Naive design of state FB

- This naive design is usually doable only in a simple case (2nd ord. system).
- Consider a system (pendulum with freq. ω_0) and place a double pole to s = -2ω_0 (it doubles the natural frequency and increases damping from 0 to 1) .

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$p(s) = \left(s + 2\omega_0\right)^2 = s^2 + 4\omega_0 s + 4\omega_0^2$$

- We require the characteristic polynomial of the resulting system to be
- The generally written characteristic polynomial is

$$\det\left(s\mathbf{I} - \left(\mathbf{A} - \mathbf{BK}\right)\right) = \det\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\right\} = \det\begin{bmatrix} s & -1 \\ k_1 + \omega_0^2 & s + k_2 \end{bmatrix}$$

$$= s^2 + k_2 s + k_1 + \omega_0^2$$

- Comparing the general and required coefficients of SF we obtain

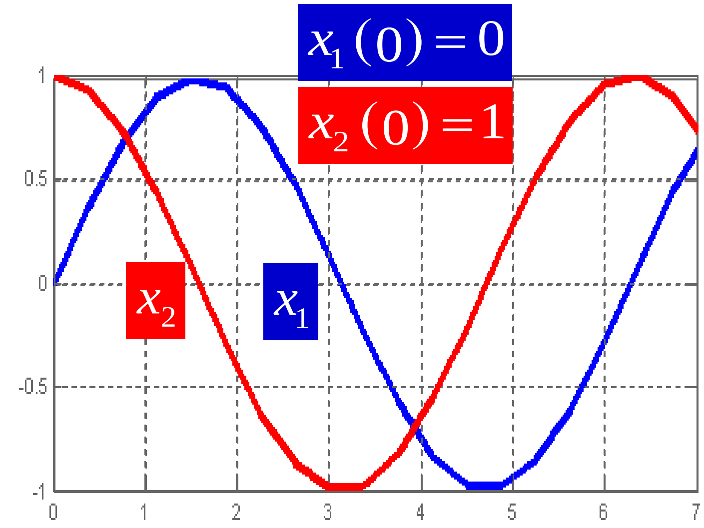
$$\begin{aligned} k_1 + \omega_0^2 &= 4\omega_0^2 \\ k_2 &= 4\omega_0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} k_1 &= 3\omega_0^2 \\ k_2 &= 4\omega_0 \end{aligned} \quad \Rightarrow \quad \mathbf{K} = \begin{bmatrix} 3\omega_0^2 & 4\omega_0 \end{bmatrix}$$

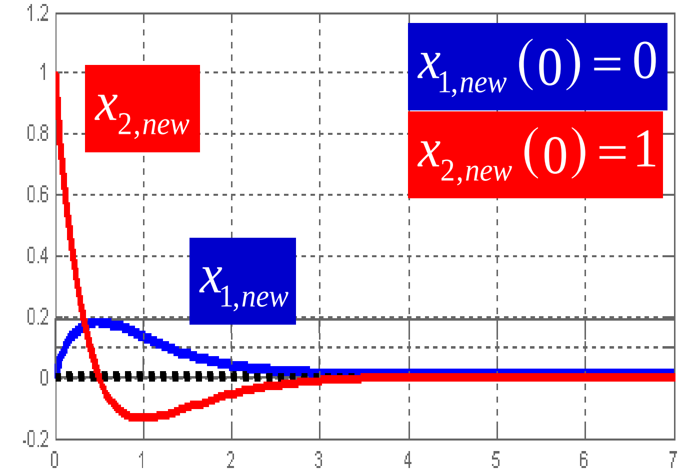
Example: Naive design of state FB

- In Matlab for $\omega_0 = 1$

```
>> omega0=1; A=[0 1;-omega0^2 0];…
    B=[0; 1];C=eye(2); D=[0;0]; pend=ss(A,B,C,D)
A =
         x1   x2
    x1    0    1
    x2   -1    0
B =
         u1
    x1    0
    x2    1
C =
         x1   x2
    y1    1    0
    y2    0    1
D =
         u1
    y1    0
    y2    0

>> K=[3*omega0^2 4*omega0]
K =   3      4
>> Anew=A-B*K
Anew =
      0      1
     -4     -4
>> pend_FB=ss(Anew,B,C,D);
>> impulse(pend,pend_FB,7)
```





Example: State FB in special form

- Movement dynamics of the monsters' foot in Jurassic Park (BiDo 11ed P11.16): Model is in nominal controller form, the goal is to get poles to $s_{1,2} = -2 \pm 2j$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

- Closing the state FB loop we receive the state matrix of CL as

$$\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -2-k_1 & -k_2 \\ 1 & 0 \end{bmatrix}$$

- Its characteristic polynomial is $p_{new}(s) = s^2 + (2+k_1)s + k_2$
- The required characteristic polynomial is

$$p_{CL}(s) = (s+2+2j)(s+2-2j) = s^2 + 4s + 8$$

- By comparison of the polynomials we obtain $k_1 = 2, k_2 = 8$
- Verification:

$$\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -8 \\ 1 & 0 \end{bmatrix}$$

Transformation matrix for conversion into a can. form.

Clear but longer procedure - calculation $\mathbf{T}$: $\mathbf{x} = \mathbf{T}\mathbf{x}_{\text{con}}, \mathbf{B} = \mathbf{T}\mathbf{B}_{\text{con}}, \mathbf{A} = \mathbf{T}\mathbf{A}_{\text{con}}\mathbf{T}^{-1}$

1. Calculate controllability matrix $\mathcal{C}_x = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$
2. Obtain a canonical form (we calculate the characteristic polynomial and substitute the coefficients into known matrix structures $\mathbf{A}_{\text{con}}, \mathbf{B}_{\text{con}}$)
3. Calculate controllability matrix $\mathcal{C}_{\text{con}} = \begin{bmatrix} \mathbf{B}_{\text{con}} & \mathbf{A}_{\text{con}}\mathbf{B}_{\text{con}} & \dots & \mathbf{A}_{\text{con}}^{n-1}\mathbf{B}_{\text{con}} \end{bmatrix}$ in new coordinates
4. Calculate the transformation matrix $\boxed{\mathbf{T} = \mathcal{C}_x \mathcal{C}_{\text{con}}^{-1}}$

More complex but shorter procedure - calculation: $\mathbf{T}^{-1}$

1. As above $\mathcal{C}_x = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$
2. Calculate the last row of the matrix $\mathbf{T}^{-1}$ as $\mathbf{t}_n = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix} \mathcal{C}_x^{-1}$
3. Calculate the matrix

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{t}_n \mathbf{A}^{n-1} \\ \mathbf{t}_n \mathbf{A}^{n-2} \\ \vdots \\ \mathbf{t}_n \end{bmatrix}$$

System has to be controllable, otherwise it does not work

Explanation of other procedure

- The procedure uses a special form of matrices $\mathbf{A}_{\text{con}}$, $\mathbf{B}_{\text{con}}$
- We explain it on a 3rd order system. Generally, it is the same.
- Split the matrix $\mathbf{T}^{-1}$ into rows denoted as follows $\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix}$
- Since

$$\mathbf{A}_{\text{con}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \rightarrow \mathbf{A}_{\text{con}}\mathbf{T}^{-1} = \mathbf{T}^{-1}\mathbf{A}$$

it holds, that

$$\begin{bmatrix} -a_2/a_3 & -a_1/a_3 & -a_0/a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix}\mathbf{A} \rightarrow \begin{aligned} \mathbf{t}_2 &= \mathbf{t}_3\mathbf{A} \\ \mathbf{t}_1 &= \mathbf{t}_2\mathbf{A} = \mathbf{t}_3\mathbf{A}^2 \end{aligned}$$

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{t}_3\mathbf{A}^2 \\ \mathbf{t}_3\mathbf{A} \\ \mathbf{t}_3 \end{bmatrix}$$

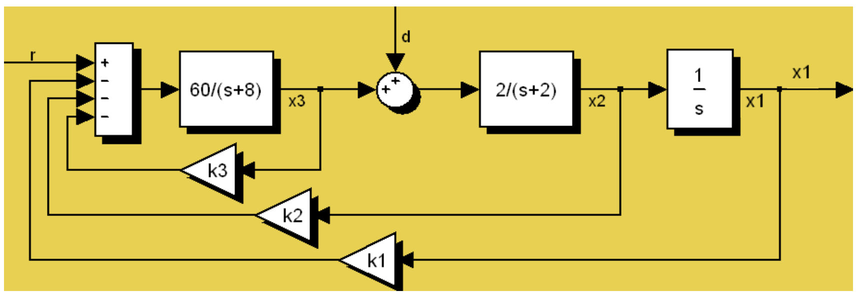
$$\mathbf{B}_{\text{con}} = \mathbf{T}^{-1}\mathbf{B} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix}\mathbf{B} \rightarrow \begin{aligned} \mathbf{t}_3\mathbf{B} &= 0 \\ \mathbf{t}_2\mathbf{B} &= 0 = \mathbf{t}_3\mathbf{A}\mathbf{B} \\ \mathbf{t}_1\mathbf{B} &= 1 = \mathbf{t}_3\mathbf{A}^2\mathbf{B} \end{aligned}$$

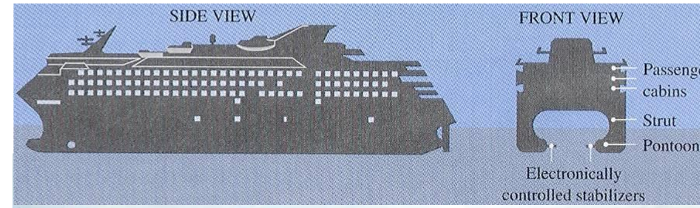
$$\rightarrow \mathbf{t}_3\begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \mathbf{t}_3\mathcal{C}_x = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \rightarrow \mathbf{t}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathcal{C}_x^{-1}$$

Example: General case

Zadání: Asia Star Cruise Ship

- reduction of side oscillations (roll)
- uses floats and stabilizers
  and a control system in the figure





SWATH = Small-Waterplane-Area-Twin-Hull

- From the diagram, it follows that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 60 \end{bmatrix} r + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} d$$

- Find state FB, such that the resulting system has poles $s_{1,2} = -2 \pm j2, s_3 = -15$

Solution using a special form

- Given data

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & -2 & 2\\ 0 & 0 & -8\end{bmatrix},\mathbf{B}=\begin{bmatrix}0\\ 0\\ 60\end{bmatrix}$$

$$p_{CL}(s)=(s+2+j2)(s+2-j2)(s+15)$$
$$=s^3+19s^2+68s+120$$

- The characteristic polynomial is

$$a(s)=s^3+10s^2+16s=s(s+2)(s+8)$$

- The matrix in canonical controllability form can be written as

$$\mathbf{A}_{con}=\begin{bmatrix}-10 & -16 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\mathbf{B}_{con}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

```
>> pformat rootc
>> A=[0 1 0;0 -2 2; 0 0 -8],B=[0;0;60]
A =
     0     1     0
     0    -2     2
     0     0    -8
B =
     0
     0
    60
>> a=det(s*eye(3)-A)
a =   s(s+8)(s+2)
>> s1=-2+j*2;s2=s1';s3=-15;
>> pcl=(s-s1)*(s-s2)*(s-s3)
pcl =  (s+2+2i)(s+2-2i)(s+15)
```

```
 >> Acon=[-a{2},-a{1},-a{0};1,0,0;0,1,0],
    Bcon=[1;0;0]
Acon =
   -10   -16     0
     1     0     0
     0     1     0
Bcon =
     1
     0
     0
```

Solution using a special form

Calculation of transformation matrix

1. By use of a canonical form

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 120 \\ 0 & 120 & -1200 \\ 60 & -480 & 3840 \end{bmatrix}$$

$$\mathcal{C}_{con} = \begin{bmatrix} 1 & -10 & 84 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 120 \\ 0 & 120 & 0 \\ 60 & 120 & 0 \end{bmatrix}$$

$$\mathbf{T} = \mathcal{C}_x \mathcal{C}_{\mathrm{con}}^{-1}$$

```
>> CON=[B A*B A*A*B]
CON =
     0     0   120
     0   120 -1200
    60  -480  3840
>> CONcon=[Bcon,Acon*Bcon,Acon*Acon*Bcon]
CONcon =
     1   -10    84
     0     1   -10
     0     0     1
>> T=CON/CONcon
T =
     0     0   120
     0   120     0
    60   120     0
```

2. or a second method in which T is calculated first and then the canonical form.

```
>> CONT=[B A*B A*A*B];CONTi=inv(CONT);
>> t3=CONTi(3,:);
>> Ti=[t3*A^2;t3*A;t3];T=inv(Ti);
>> Acon=Ti*A*T,Bcon=Ti*B;
```

Example: general case using the special form

- Design in canonical form

$$\mathbf{K}_{\text{con}} = \begin{bmatrix} 9 & 5 & 120 \end{bmatrix}$$

$$\mathbf{A}_{\text{new,con}} = \begin{bmatrix} -19 & -68 & -120 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

```
>> Kcon=pcl{2:-1:0}-a{2:-1:0}
Kcon = 9     5   120
>> Anewcon=Acon-Bcon*Kcon
Anewcon =
    19    -68    120
     1      0      0
     0      1      0
eig(Anewcon)
ans =
  -15 + 0i
   -2 + 2i
   -2 - 2i
```

- Feedback vector transformation into original coordinates

$$\mathbf{K} = \mathbf{K}_{con}\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0.2833 & 0.1500 \end{bmatrix}$$

- Verification

$$\mathbf{A}_{\text{new}} = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ -60 & -17 & -17 \end{bmatrix}$$

$$\det\left(s\mathbf{I} - \mathbf{A}_{\text{new}}\right) = (s+15)(s+2+j2)(s+2-j2)$$

```
>> >> K=Kcon/T
K =  1.0000     0.2833     0.1500
>> Anew=A-B*K
Anew =
     0      1      0
     0     -2      2
   -60    -17    -17
>> eig(Anew)
ans =
  -15 + 0i
   -2 + 2i
   -2 - 2i
```

Example: general case by Ackermann's formula

- Calculation by use of the Ackermann's formula

$$\mathbf{K} = \begin{bmatrix} 1 & 0.2830 & 0.1500 \end{bmatrix}$$

```
>> KK=CON\mvalue(pcl,A)
KK =
   16.0000    8.0000    2.0000
   10.0000    3.2667    0.8667
    1.0000    0.2833    0.1500
>> KK=KK(3,:)
KK =
    1.0000    0.2833    0.1500
```

- In CSTbx, there are special functions **`acker`** and **`place`**

```
>> KKK=acker(A,B,[s1,s2,s3])
KKK =
    1.0000    0.2833    0.1500
>> KKKK=place(A,B,[s1,s2,s3])
KKKK =
    1.0000    0.2833    0.1500
```

Example: Thermal system

Plant (Fe5s479)

$$\mathbf{A}=\begin{bmatrix}-7 & 1\\ -12 & 0\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}1\\ -z_0\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}1 & 0\end{bmatrix}\qquad D=0$$

```
>> syms s z0 k1 k2 dzeta omegan
>> A=[-7 1;-12 0],B=[1;-z0],C=[1 0],
   D=0,K=[k1 k2]
>> G=C/(s*eye(2)-A)*B+D;G=factor(G)
G =  (s-z0)/(s+4)/(s+3)
```

$$G(s)=\frac{s-z_0}{(s+4)(s+3)}$$

Analysis

- zero in $s = z_0$
- poles in s_1 = -3, s_2 = -4,
- uncontrollable for z_0= -3, -4

```
>> Cont=[B A*B]
Cont =
[     1, -7-z0]
[   -z0,   -12]
>> det(Cont)
ans =
-12-7*z0-z0^2
>> factor(det(Cont))
ans =
-(z0+4)*(z0+3)
```

Design specification

- place poles to

$$s_{1,2}=-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

- i.e. required CL characteristic polynomial

$$p_{CL}=s^2+2\zeta\omega_n s+\omega_n^2$$

```
>> pCL=s^2+2*dzeta*omegan*s+omegan^2
pCL =
s^2+2*dzeta*omegan*s+omegan^2
```

Example: Thermal system


```
>> pnew=det(s*eye(2)-(A-B*K));pCL=collect(pCL,s)
pnew=s^2+(k1-z0*k2+7)*s+12-z0*k1-7*z0*k2-12*k2
>> roz=collect(pCL-pCL,s)
roz = (k1-z0*k2+7-2*dzeta*omegan)*s
         -12*k2+12-z0*k1-7*z0*k2-omegan^2
```

Design – by naive method

- CL characteristic polynomial

$$p_{new} = \det\left(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})\right)$$
$$= s^2 + (k_1 - z_0 k_2 + 7)s + 12 - k_1 z_0 - k_2(7z_0 + 12)$$

- By comparison with the requirement

$$p_{CL} = s^2 + 2\zeta\omega_n s + \omega_n^2$$

- we obtain following equations

$$k_1 - z_0 k_2 = 2\zeta\omega_n - 7$$
$$-k_1 z_0 - k_2(7z_0 + 12) = \omega_n^2 - 12$$

- and from them it follows, that

$$k_2 = \frac{z_0(7 - 2\zeta\omega_n) + 12 - \omega_n^2}{(z_0+3)(z_0+4)}$$

$$k_1 = \frac{z_0(14\zeta\omega_n - 37 - \omega_n^2) + 12(2\zeta\omega_n - 7)}{(z_0+3)(z_0+4)}$$

```
>> KK=solve('k1-z0*k2+7-2*dzeta*omegan',…
 '-12*k2+12-z0*k1-7*z0*k2-omegan^2',k1,k2)
>> [NK1,DK1] = numden(KK.k1)
>> [NK2,DK2] = numden(KK.k2)
>> Nk1=collect(NK1,z0)
>> Nk2=collect(NK2,z0)
>> Dk=factor(DK2)
>> k1=Nk1/Dk
k1=
((-37-omegan^2+14*dzeta*omegan)*z0-84
 +24*dzeta*omegan) /(z0+4)/(z0+3)
>> k2=Nk2/Dk
k2=((7-2*dzeta*omegan)*z0-omegan^2+12)
     /(z0+4)/(z0+3)
```

Result analysis

- Coefficients of state FB are

$$k_1 = \frac{z_0(14\zeta\omega_n - 37 - \omega_n^2) + 12(2\zeta\omega_n - 7)}{(z_0+3)(z_0+4)}$$

$$k_2 = \frac{z_0(7 - 2\zeta\omega_n) + 12 - \omega_n^2}{(z_0+3)(z_0+4)}$$

- FB gain increases, if zero $\boldsymbol{z_0}$ approaches -3 or -4, i.e.
- if the plat looses controllability
- By loosing the controllability, the controller produces more effort

- FB gain increases also with rising $\boldsymbol{\omega_n}$
- It shows that, if we want to move poles away from the original values, we have to use large gain

2 examples: State FB doesn't touch zeros

- Asia Star

```
>> A=[0 1 0;0 -2 2;0 0 -8];
B=[0;0;60]; C=[1 0 0];
>> G=sdf(A,B,C)
G =
        1.2e+002
    -----------------
    16s + 10s^2 + s^3
>> KKK=place(A,B,[-2+2*j,-2-2*j,-15])
KKK =
    1.0000    0.2833    0.1500
>> Anew=A-B*KKK
Anew =
         0    1.0000         0
         0   -2.0000    2.0000
  -60.0000  -17.0000  -17.0000
>> Gnew=sdf(Anew,B,C)
Gnew =
              1.2e+002
    ----------------------------
    1.2e+002 + 68s + 19s^2 + s^3
```

- Thermal system

```
>> A=[-7 1;-12 0];B=[1;-2];|
C=[1 0]; G=sdf(A,B,C)
G =
       -2 + s
    -------------
    12 + 7s + s^2
>> p=roots([1 2 4])
p =
  -1.0000 + 1.7321i
  -1.0000 - 1.7321i
>> K=place(A,B,p)
K = -3.8000    0.6000
>> Anew=A-B*K
Anew =
   -3.2000    0.4000
  -19.6000    1.2000
>> Gnew=sdf(Anew,B,C)
Gnew =
       -2 + s
    ------------
    4 + 2s + s^2
```

- In both cases, the zeros didn't change, appear or disappear but poles yes.

The reduction of zeros - How to use it?

- Plant

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u,\ y = \begin{bmatrix} 0 & 20 & 100 \end{bmatrix}\mathbf{x}$$

with a transfer function

$$G(s) = \frac{20(s+5)}{s(s+1)(s+4)}$$

- Specification

$$\%OS = 9.5\%, T_s = 0.74\text{s} \rightarrow \text{s}_{1,2} = -5.4 \pm j7.2$$

- The third pole is chosen to cancel the system zero $\text{s}_3 = -5$

- Design $K = \begin{bmatrix} 10.8 & 131 & 405 \end{bmatrix}$

$$A_{celk} = \begin{bmatrix} -15.8 & -135 & -405 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Resulting transfer function

$$T_{celk}(s) = \frac{20}{s^2 + 10.8s + 81} = \frac{20}{(s+5.4+j7.2)(s+5.4-j7.2)}$$

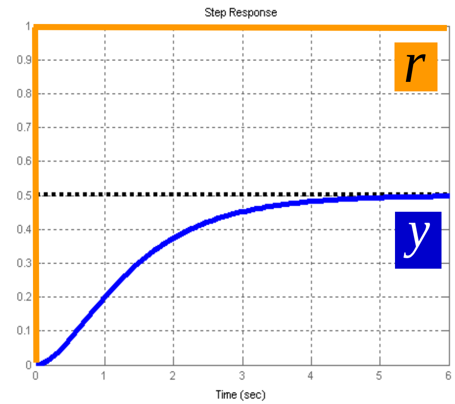
- If the system does not have zeros (there is nothing to cancel), we choose other CL poles 'sufficiently far to left'

Example: Steady-state response

- System with matrices $\mathbf{A}=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix},\mathbf{B}=\begin{bmatrix}0\\ 1\end{bmatrix},\mathbf{C}=\begin{bmatrix}1 & 0\end{bmatrix}$

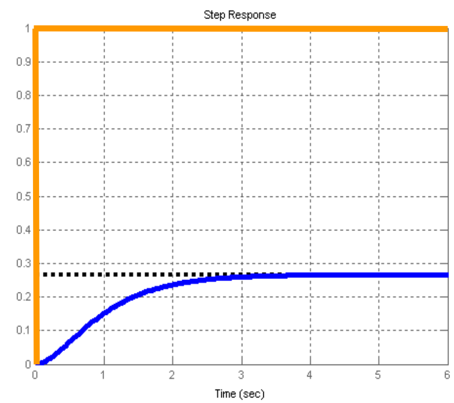
- Placing poles to -1,-2 → $K_1=\begin{bmatrix}2 & 2\end{bmatrix}$

```
>> A=[0 1;0 -1];B=[0;1];C=[1 0];J=0;
>> Ka=place(A,B,[-1,-2])
K1 = 2.0000    2.0000
>> G1=sdf(A-B*K1,B,C), step(G1)
G1 =      1
     ------------
     2 + 3s + s^2
```



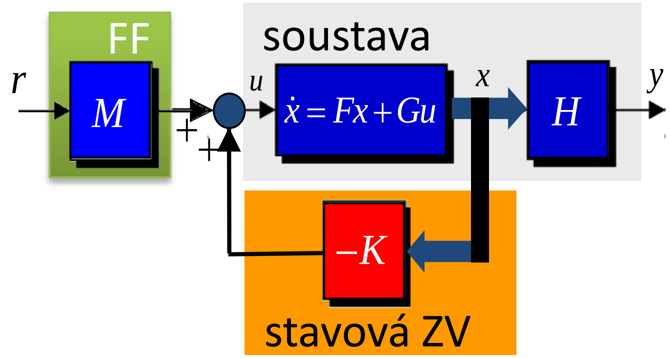
- Placing poles to -1.5,-2.5 → $K_2=\begin{bmatrix}3.75 & 3\end{bmatrix}$

```
>> K2=place(A,B,[-1.5,-2.5])
K2 =  3.7500    3.0000
>> G2=sdf(A-B*K2,B,C), step(G2)
fb =
           1
     --------------
     3.7 + 4s + s^2
```



Example: Solution by FF

- The steady-state gain can be „fixed“ by a feed-forward term.



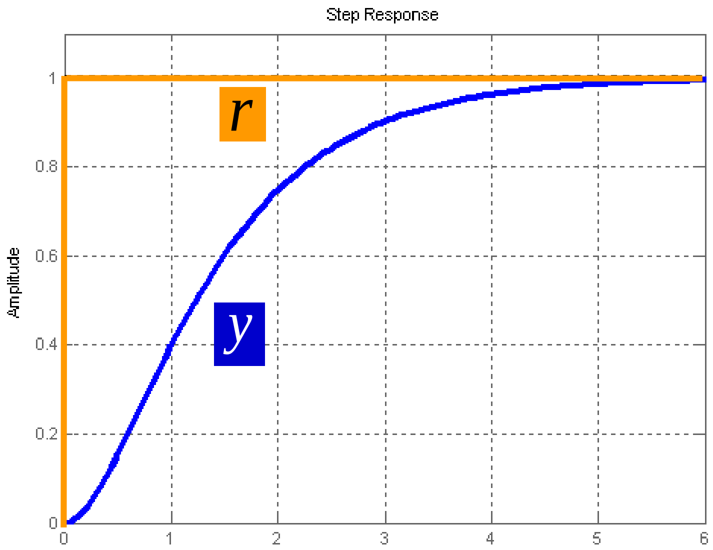
$$G(0) = -\mathbf{C}\left(\mathbf{A} - \mathbf{BK}\right)^{-1}\mathbf{B}$$

$$M = 1/G(0)$$

$$F(0) = G(0)M = 1$$

- Continued example

```
>> G1=rdf(G1)
G1 = 1 / 2 + 3s + s^2
>> M=1/value(G1,0)
M =   2.0000
>> F1=M*G1
F1 = 2 /  2 + 3s + s^2
>> step(F1)
>> MM=-1/(C/(A-B*K1)*B)
MM =  2.0000
```



Example: Integral control

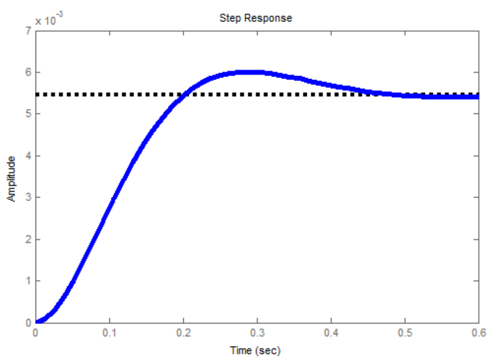
- For a system $\dot{\mathbf{x}} = \begin{bmatrix} -5 & -3 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{x}$
- First, design
  a state feedback to ensure $OS\% = 10\%, T_S = 0.5\text{s}$
  and then design an integral control.

State FB

- The requirements are satisfied by a char. polynomial $s^2 + 16s + 183.1$
- Model is in canonical form, thus after applying FB, char. pol. will be

$$s^2 + (5 + k_1)s + (3 + k_2)$$

- By comparison we obtain $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 11 & 180.1 \end{bmatrix}$
- After applying FB



$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -16 & -183.1 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$

$$y = \mathbf{C}\mathbf{x} = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{x}$$

but

$$e_{step,ss} = 1 + \mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B} = 1 - 0.0055 = 0.9945$$

Example: Integral control

Integral control

- By adding an integrator we obtain a 3rd order system.
- Because the system doesn't have zeros, we add to characteristic polynomial a pole „far to left"

$$\left(s^2+16s+183.1\right)\left(s+100\right)=s^3+116s^2+1783.1s+18310$$

- The characteristic polynomial of the final system (see next page) is

$$s^3+\left(5+k_1\right)s^2+\left(3+k_2\right)s-K_I$$

- By comparison we obtain

$$k_1=111, k_2=1780.1, K_I=-18310$$

- Alternatively, we can also proceed by Ackermann's formula for (overally) large systems

$$\mathbf{K}_{celk}=\begin{bmatrix}\mathbf{K} & K_I\end{bmatrix}=\begin{bmatrix}111 & 1780.1 & -18310\end{bmatrix}$$

Example: Integral control

The system is in canonical form, therefore the solution is simple:

- Final system with all feedbacks is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_I\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}-5 & -3\\ 1 & 0\end{bmatrix}-\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix} & -\begin{bmatrix}1\\ 0\end{bmatrix}K_I\\ -\begin{bmatrix}0 & 1\end{bmatrix} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_I\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r$$

$$=\begin{bmatrix}-(k_1+5) & -(k_2+3) & -K_I\\ 1 & 0 & 0\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_I\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r,\quad y=\begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_I\end{bmatrix}$$

- Its characteristic polynomial is

$$\det\left(s\mathbf{I}-\mathbf{A}_{big}\right)=\det\begin{bmatrix}s+(k_1+5) & (k_2+3) & K_I\\ -1 & s & 0\\ 0 & 1 & s\end{bmatrix}$$

$$=\left(s+k_1+5\right)s^2+\left(k_2+3\right)s-K_I=s^3+\left(5+k_1\right)s^2+\left(3+k_2\right)s-K_I$$

Example: Integral control

- Calculation by Ackermann's formula

```
>> A=[-5 -3;1 0];B=[1;0];C=[0 1]; Abig=[A,zeros(2,1);-C,0],Bbig=[B;0]
Abig =
    -5    -3     0
     1     0     0
     0    -1     0
Bbig =
     1
     0
     0
>> CON=[Bbig,Abig*Bbig,Abig^2*Bbig]
CON =
     1    -5    22
     0     1    -5
     0     0    -1
>> p_celk=(s+100)*(s^2+16*s+183.1)
p_celk = 1.8e+004 + 1.8e+003s + 1.2e+002s^2 + s^3
>> roots(p_celk)
ans =
  -8.0000 +10.9133i
  -8.0000 -10.9133i
  -1.0000
>> Kcelk=[0 0 1]*inv(CON)*mvalue(p_celk,Abig)
Kcelk =  1.0e+004 *
    0.0111    0.1780   -1.8310
```

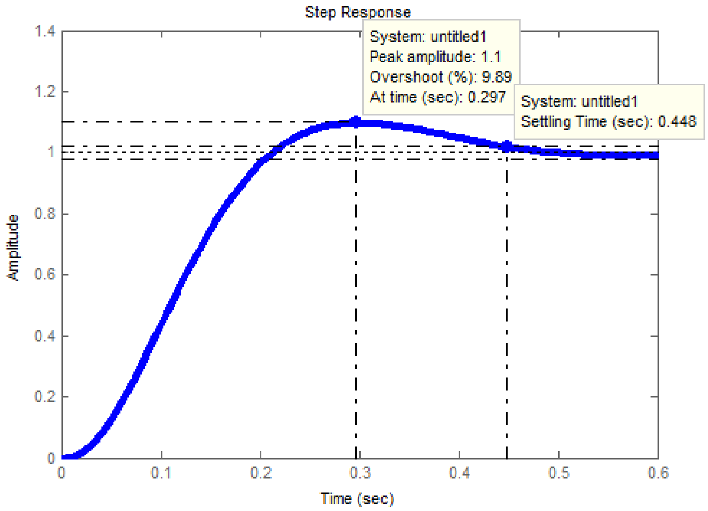
```
>> poles=roots(p_celk)
poles =
  1.0e+002 *
  -1.0000
  -0.0800 + 0.1091i
  -0.0800 - 0.1091i
>> place(Abig,Bbig,poles)
ans =
  1.0e+004 *
    0.0111    0.1780   -1.8310
```

Example: Integral control

- Simulation

```
>> Cbig=[C 0],Bref=[0;0;1]
Cbig =   0     1     0
Bref =
     0
     0
     1
>> T=sdf((Abig-Bbig*Kcelk),Bref,Cbig)
                1.8310e+004
 ------------------------------------------
1.8310e+004 + 1.7831e+003s + 116.0000s^2 + s^3
>> step(T)

>> Tdc=value(T,0)
Tdc = 1
>> -Cbig*inv(Abig-Bbig*Kcelk)*Bref
ans = 1.0000
>> e_infty=1+Cbig*inv(Abig-Bbig*Kcelk)*Bref
e_infty = 1.1102e-016
```



$$T(s) = \frac{18310}{s^3 + 116s^2 + 1783.1s + 18310}$$

$$e_{step,ss} = 1 + \mathbf{C}_{big}\left(\mathbf{A}_{big} - \mathbf{B}_{big} K_{celk}\right)^{-1}\mathbf{B}_{ref} = 1 - 1 = 0$$