Exercises for lectures
20 – Digital Control

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Automatic control 2016

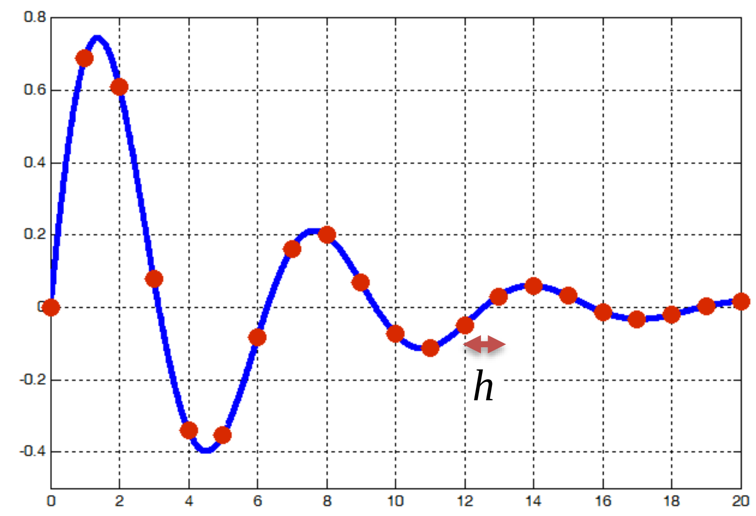
Sampling: *s* and *z* relationship for complex poles

Continuous signal $y(t)=e^{-\alpha t}\sin\beta t, t>0$

- Laplace transform

$$y(s)=\frac{\beta}{(s+\alpha)^2+\beta^2}$$

with poles $s_{1,2}=-\alpha\pm j\beta$



Discrete signal

- z-Transform $y(k)=e^{-\alpha kh}\sin(\beta kh)$

$$y(z)=\frac{z^{-1}e^{-\alpha h}\sin(\beta h)}{1-z^{-1}2e^{-\alpha h}\cos(\beta h)+z^{-2}e^{-2\alpha h}}=\frac{ze^{-\alpha h}\sin(\beta h)}{z^2-z2e^{-\alpha h}\cos(\beta h)+e^{-2\alpha h}}$$

- With poles $z_{1,2}=e^{-\alpha h}\left(\cos(\beta h)\pm j\sin(\beta h)\right)=e^{(-\alpha\pm j\beta)h}$

- There is a relationship between the continuous and sampled system poles.

$$z_{1,2}=e^{s_{1,2}h}$$

Stroboscopic effect

- Signals with frequencies from 0 to π/h are displayed to unit circle after sampling. Where are displayed signals with higher frequencies?
- Consider sine signal $y(t)=\sin\omega_1 t$ with L-transform $y(s)=\dfrac{\omega_1}{s^2+\omega_1^2}$ and poles $s_{1,2}=\pm j\omega_1$
- The sampling period *h*

$$y(k)=\sin(\omega_1 hk) \qquad y(z)=\frac{z\sin\omega_1 h}{z^2-z2\cos\omega_1 h+1} \qquad z_{1,2}=e^{\pm j\omega_1 h}$$

- In case

$$\omega_1>\pi/h \leftrightarrow h>\pi/\omega_1$$

in Hz

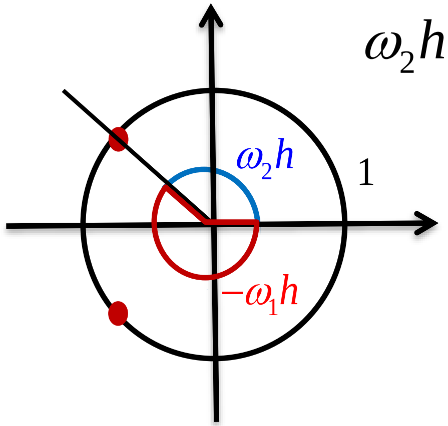
$$f_1=\frac{\omega_1}{2\pi}\ \text{Hz},\ \ f_s=\frac{1}{h}<2f_1$$

discrete poles have an angle > 180° $\omega_1 h>\pi$

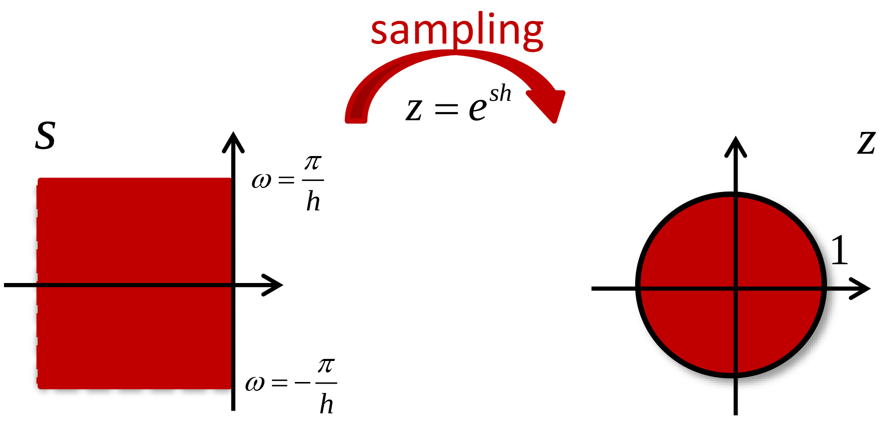
- for $\omega_1 h>\pi$ is $e^{-j\omega_1 h}=e^{j(2\pi-\omega_1 h)},\ e^{j\omega_1 h}=e^{-j(2\pi-\omega_1 h)}$, where $(2\pi-\omega_1 h)\in[0,180°]$
- And position of the poles corresponds to the frequency

$$(\omega_2 h=2\pi-\omega_1 h)\rightarrow \quad \omega_2=2\pi/h-\omega_1=\omega_s-\omega_1$$

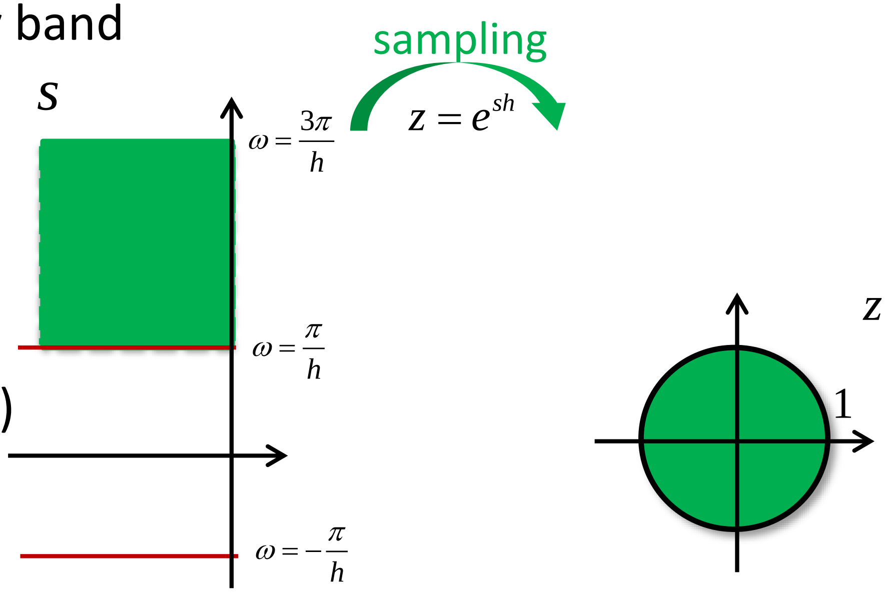
Stroboscopic effect



$$\omega_2 h = 2\pi - \omega_1 h$$



- We don't know the correct frequency band in a reverse transform (signal reconstruction)
- To prevent this, we have to sample with a higher sample rate.
  Or filter out a higher frequencies than $\omega_N = \omega_s / 2$ (anti-aliasing filter)



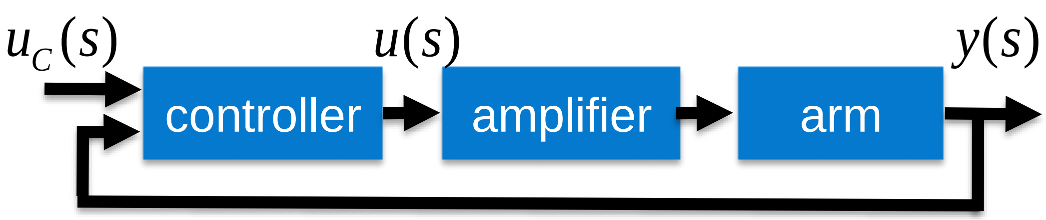
Example: Disk drive

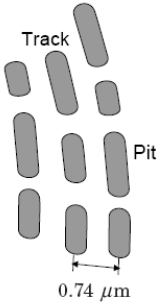
Disk drive Arm

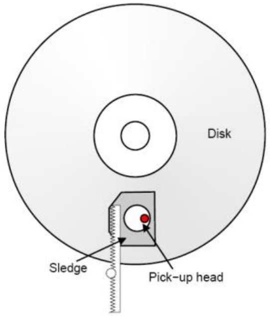
- simplified (normalized to 1)
  more detail ÅW, s13, ex1.2
- Transfer function from voltage to arm position

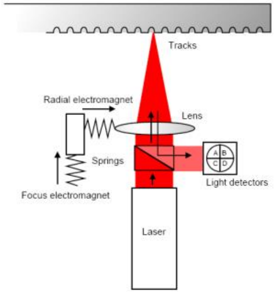
$$G(s)=\frac{1}{s^2}$$

- goal: follow the track
- Accurate position control
- Important dynamics – reading speed
- Control structure









Example: continuous control

- Continuous controller (designed by “continuous methods“)

$$u(s)=\frac{1}{2}u_C(s)-2\frac{s+0.5}{s+2}y(s)$$

- CL characteristic polynomial

$$c_{CL}(s)=(s+1)\left(s^2+s+1\right)$$

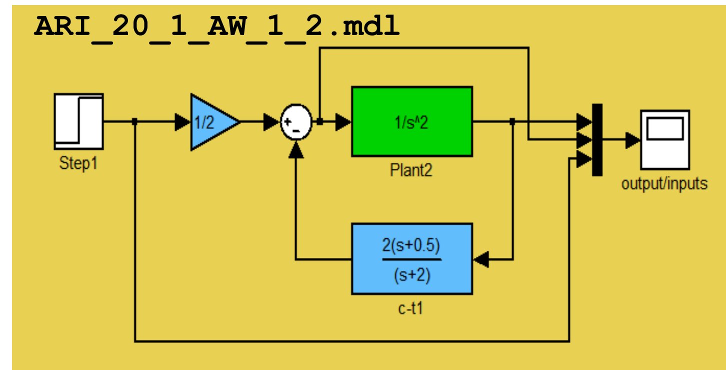
$$s_1=1$$

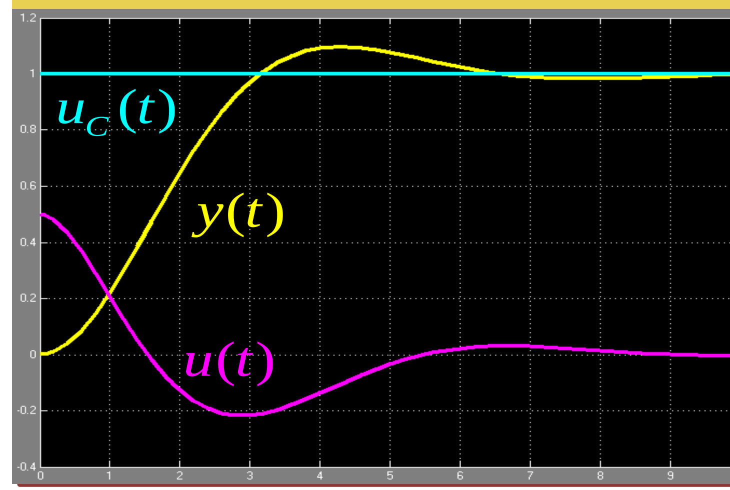
$$s_{2,3}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$$

- CL transfer function

$$y(s)=\frac{1}{2}\frac{s+2}{(s+1)\left(s^2+s+1\right)}u_C(s)$$

- simulation **`AW_1_2.mdl`**
- Setting time for 5% is 5.5, overshoot to 10% - OK
- How to realize digitally ?



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(s+1.0000)(s^2+1.0000s+1.0000)
```

Example: Naive controller approximaton

- Continuous controller is

$$\frac{s+0.5}{s+2} = 1 - \frac{1.5}{s+2}$$

$$u(s) = 0.5u_C(s) - 2\frac{s+0.5}{s+2}y(s) = 0.5u_C(s) - 2y(s) + 2\frac{1.5}{s+2}y(s)$$

$$= 2\left[0.25u_C(s) - y(s) + x(s)\right]$$

where $x(s) = \frac{1.5}{s+2}y(s)$

- We get a continuous time domain algorithm (control law)

$$u(t) = 2\left[0.25u_C(t) - y(t) + x(t)\right] \qquad \frac{dx}{dt} = -2x(t) + 1.5y(t)$$

- Discrete algorithm – we sample signal with a period *h*
- And the derivative is approximated by difference

$$\frac{x(t+h) - x(t)}{h} = -2x(t) + 1.5y(t)$$

Example: Naive design

- So we get a descrete approximation

$$x(t_k+h)=x(t_k)+h\left[1.5y(t_k)-2x(t_k)\right]$$
$$u(t_k)=2\left[0.25u_C(t_k)-y(t_k)+x(t_k)\right]$$

⇐

$$\frac{dx}{dt}=-2x(t)+1.5y(t)$$
$$u(t)=2\left[0.25u_C(t)-y(t)+x(t)\right]$$

- It can be realized by algorithm (where u_C is discrete)

```
y:= adin(in2)              {read a proces value}
u:= 2*(0.25*uc-y+x)        {compute a control value}
dout(u)                    {send out a control value}
x:= x+h(1.5*y-2*x)         {compute the new x value}
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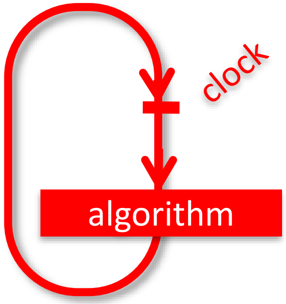


- Or discrete transfer function

$$u(z)=2\left[0.25u_C(z)-y(z)+x(z)\right]$$
$$zx(z)=x(z)+h\left[1.5y(z)-2x(z)\right]$$

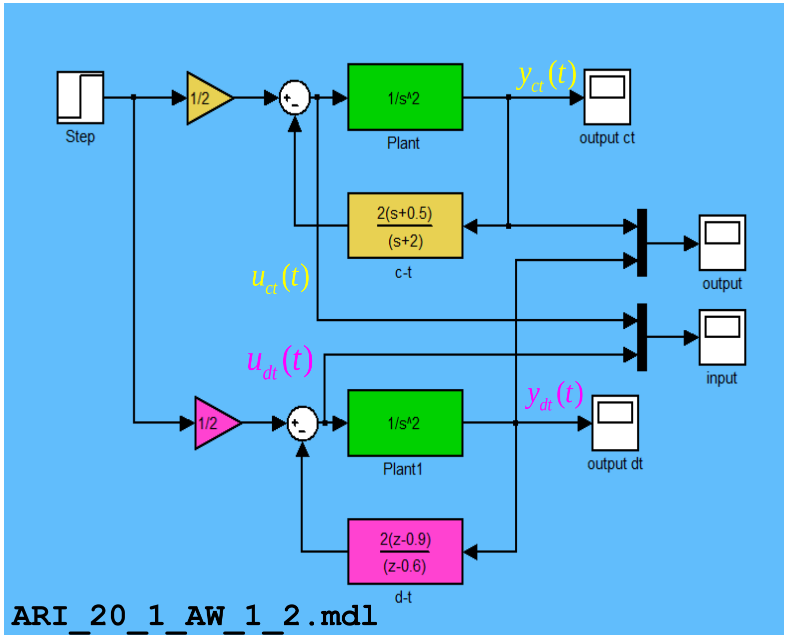
$$u(z)=0.5u_C(z)-2\frac{z+0.5h-1}{z+2h-1}y(z)$$

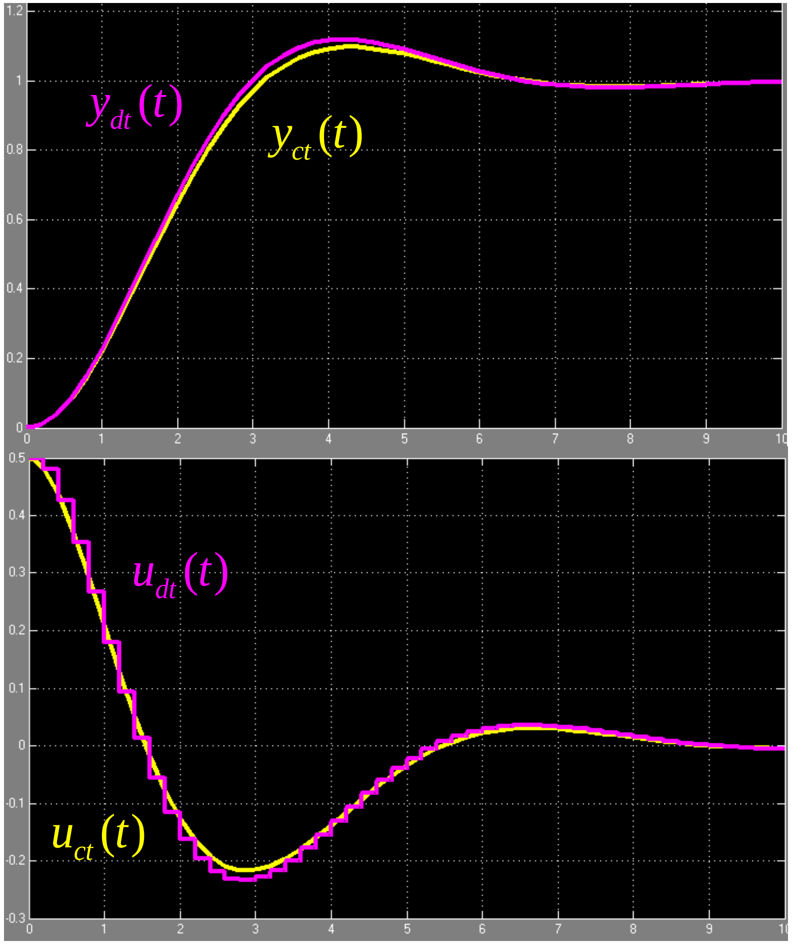
⇐ Corresponds to continuous transfer function substitution $s=\frac{z-1}{h}$

Example: comparison

- Comparin continuous and discrete control for $h = 0.2$

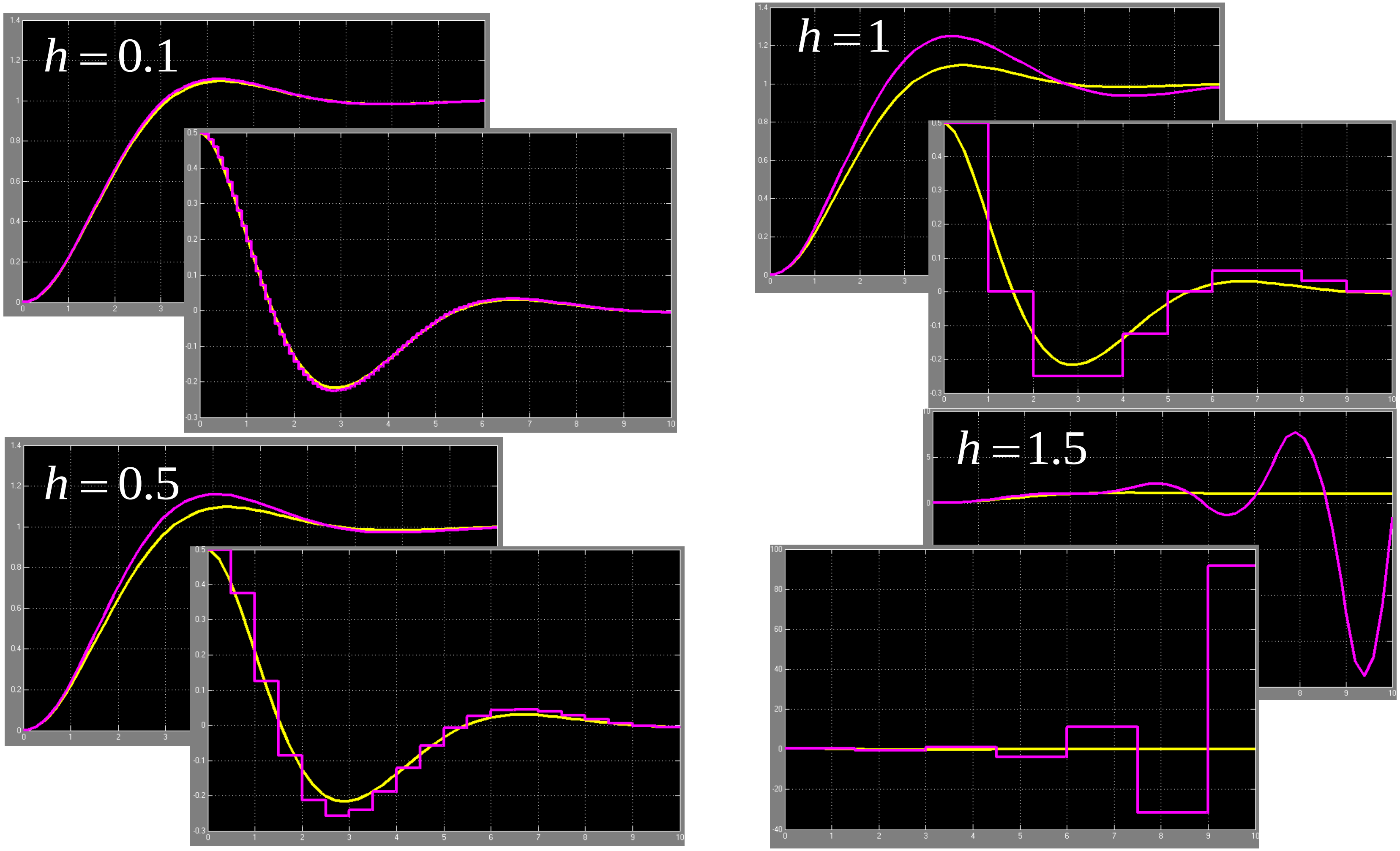
$$u(z) = 0.5u_C(z) - 2\frac{z-0.9}{z-0.6}y(z)$$





Example: comparison

- Various sampling periods h = 0.1, 0.5, 1, 1.5



Example: another solution

- We find a discrete transfer function of a system and a shaper.

$$G(s)=\frac{1}{s^2} \quad \Rightarrow \quad G(z)=\frac{h^2}{2}\frac{z+1}{(z-1)^2}$$

- We use discreté methods for descreet controller design

$$\left(z^2-2z+1\right)p(z)+h^2/2\left(z+1\right)q(z)=z^3$$

- Solve the equation

$$\begin{bmatrix} p_0 & p_1 & q_0 & q_1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ h^2/2 & h^2/2 & 0 & 0 \\ 0 & h^2/2 & h^2/2 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

- We get

$$\begin{bmatrix} p_0 & p_1 & q_0 & q_1 \end{bmatrix}=\begin{bmatrix} 3/4 & 1 & -\frac{3}{2h^2} & \frac{5}{2h^2} \end{bmatrix}$$

$$p(z)=3/4+z$$

$$q(z)=-\frac{3}{2h^2}+\frac{5}{2h^2}z$$

Example: another solution

- This „pure discrete“ controller

$$u(z) = \frac{4}{7h^2} u_C(z) - \frac{5}{2h^2} \frac{z - 3/5}{z + 3/4} y(z)$$

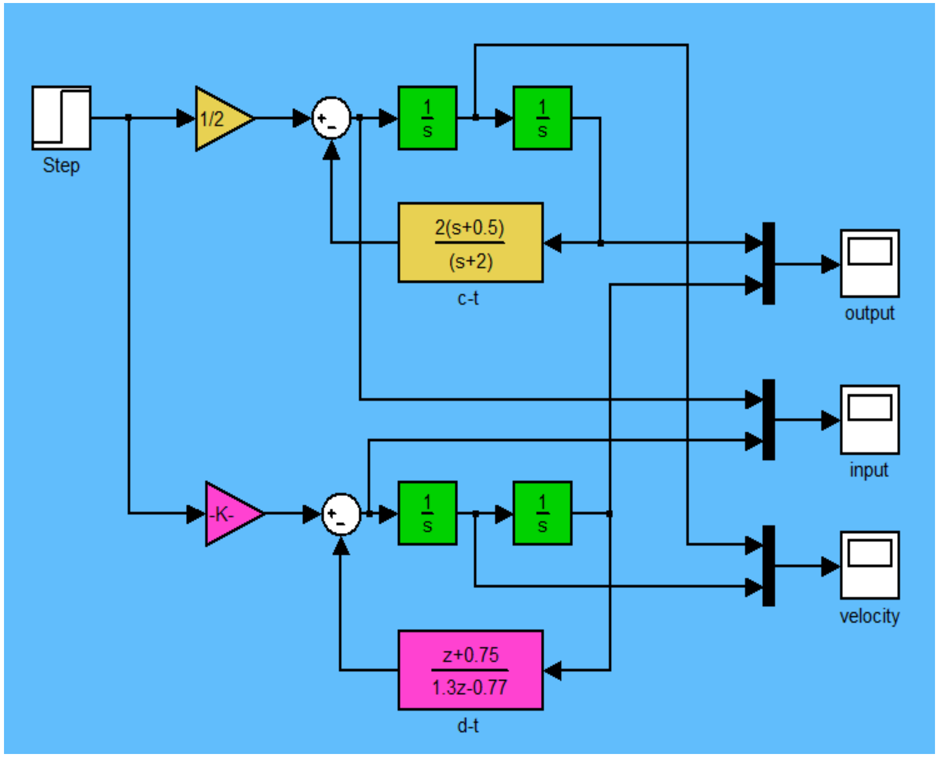
- It gives a transfer function

$$y(z) = \frac{2}{7} \frac{(z+1)(z+3/4)}{z^3} u_C(z)$$

- and CL characteristic polynomial

$$c_{CL}(z) = z^3$$

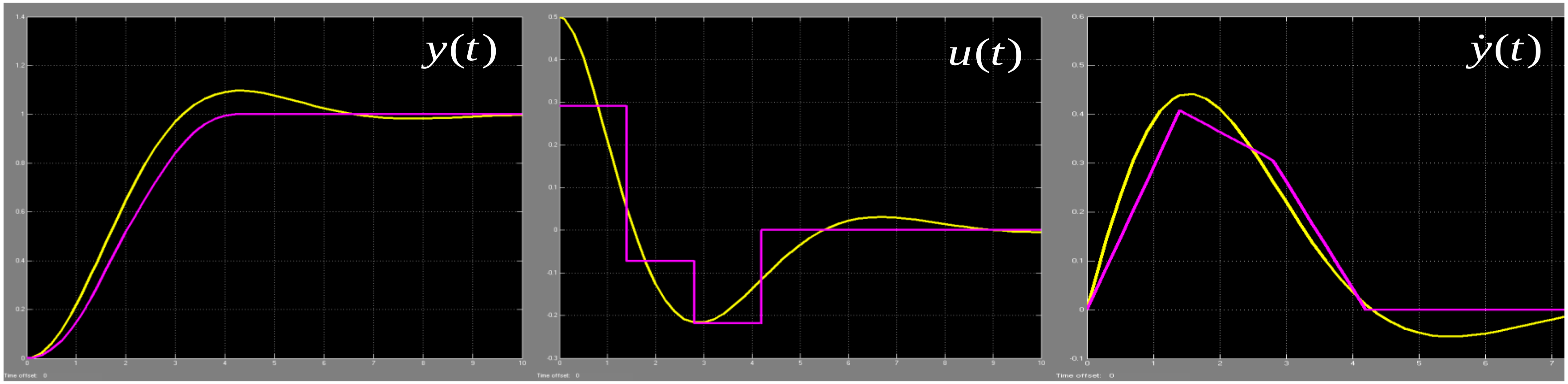
- simulation
  ARI_20_2_AW_1_3.mdl
  for $h = 1.4$



Example: another solution

- Simulation **`ARI_20_2_AW_1_3.mdl`** for $h = 1.4$

  output: input: speed:



- The output value is the same as required value in the 4th sample step.
- This pure discrete solution is better than continuous and
- There is no parallel in continuous system.
- So what happends with decreasing h ?

Example: second another solution

- Is it possible to decrease the number of steps even more? Apparently yes: $\left(z^2-2z+1\right)p(z)+h^2/2\left(z+1\right)q(z)=z^2(z+1)$
- Solve

$$\begin{bmatrix} p_0 & p_1 & q_0 & q_1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ h^2/2 & h^2/2 & 0 & 0 \\ 0 & h^2/2 & h^2/2 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$

- we get

$$\begin{bmatrix} p_0 & p_1 & q_0 & q_1 \end{bmatrix}=\begin{bmatrix} 1 & 1 & -\dfrac{2}{h^2} & \dfrac{4}{h^2} \end{bmatrix} \Rightarrow \begin{aligned} p_{weak}(z)&=1+z \\ q_{weak}(z)&=4/h^2\, z-2/h^2 \end{aligned}$$

- controller

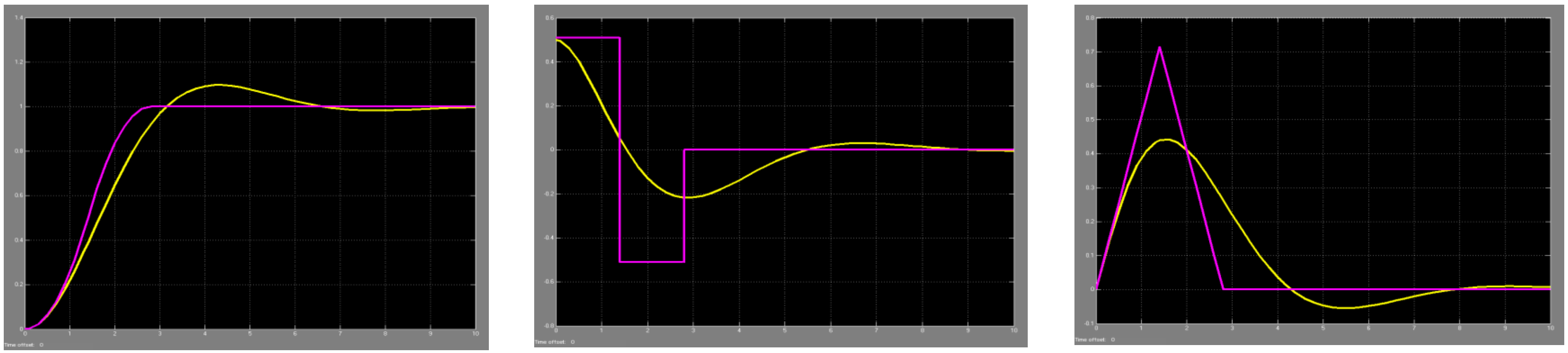
$$u=\frac{1}{h^2}u_C-\frac{4}{h^2}\frac{z-1/2}{z+1}y$$

with CL transfer function

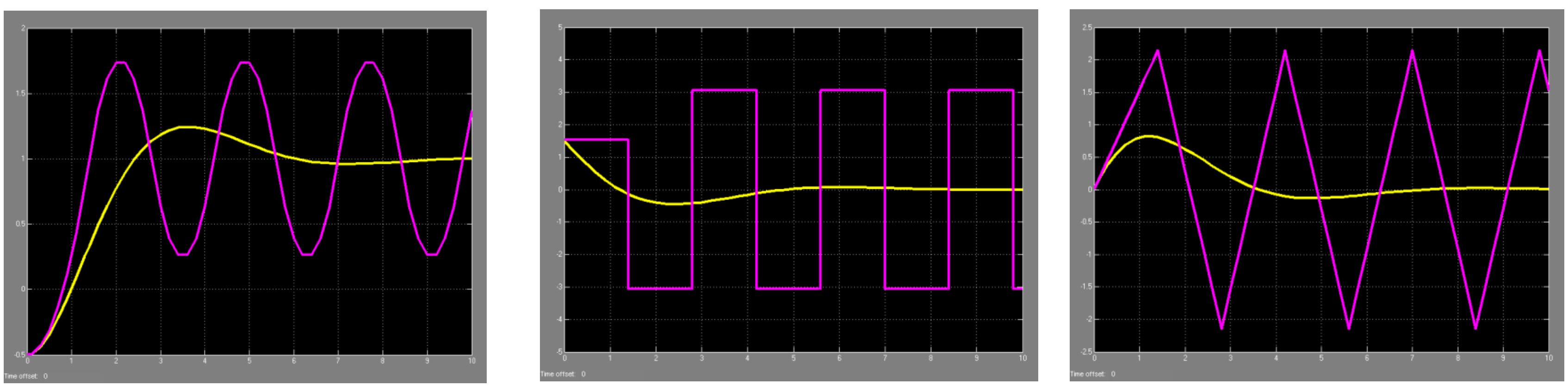
$$y=\frac{1}{2}\frac{(z+1)}{z^2}u_C$$

and CL characteristic polynomial $c_{CL}(z)=z^2(z+1)$

Simulation – second model in **ARI_20_2_AW_1_3.mdl** it looks OK



But for nonzero initial conditions reveals a problém.



Note that in moments of sampling behaves perfectly

Example: another state solution

- Solution with state feedback
- State equiations of double integrator $G(s)=1/s^2$ are

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\ 1\end{bmatrix}u(t), y=\begin{bmatrix}1 & 0\end{bmatrix}x(t)$$

- Its discrete version (with ZOH and sampling period *h*)

$$x(k+1)=\begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix}x(k)+\begin{bmatrix}h^2/2\\ h\end{bmatrix}u(k),\ y(k)=\begin{bmatrix}1 & 0\end{bmatrix}x(k)$$

- State controller

$$u(k)=-\begin{bmatrix}1/h^2\\ 3/(2h)\end{bmatrix}x(k)+1/h^2\,u_C(k)$$

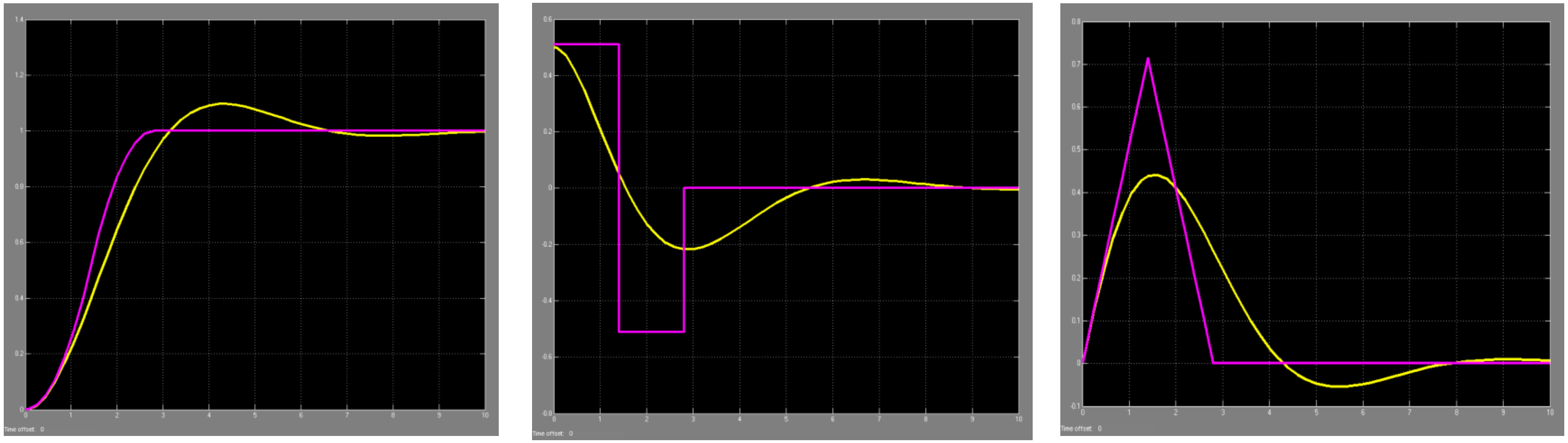
$$y(z)=\frac{1}{2}\frac{z+1}{z^2}u_C(z)$$

- The system change to with polynomial $c_{CL}(z)=z^2$

$$x(k+1)=\begin{bmatrix}1/2 & 1/4\,h\\ -1/h & -1/2\end{bmatrix}x(k)+\begin{bmatrix}1/2\\ 1/h\end{bmatrix}u_C(k),\ y(k)=\begin{bmatrix}1 & 0\end{bmatrix}x(k)$$

Simulation

- Simulation **`ARI_20_3_AW_4_5.mdl`** for $h = 1.4$



- Starting from the third step, the set point is exactly set, and control is zero.
- And for each initial condition
- system is internally stable.