Exercises for lectures
21 – Discrete-time equivalents

Michael Šebek
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Designing emulations: CL stability of cont. control does not guarantees CL stability of disc. control! We need to test "discrete stability"!
We will show it on the P regulator – simple one. We do not "approximate“ it, but use "as is" for discrete control.

- For system \( P(s) = \frac{a}{s + a} \), \( a > 0 \), regulator \( C(s) = k_p \) for cont. control

\[
CL \text{ characteristic polynomial is} \quad c_{CL}(s) = s + a + ak_p
\]

- Now, use the same controller in discrete control \( C(z) = C(s) = k_p \)
- For the discrete case analysis we use a "discrete system model“ (continuous sys. + sampler + ZOH shaper)

\[
P(z) = \frac{1 - e^{-ah}}{z - e^{-ah}}
\]

- Resulting CL characteristic polynomial is

\[
c_{CL}(z) = z - e^{-ah} + (1 - e^{-ah})k_p
\]
Introduction: CL stability for cont. and disc. control

- Continuous case \( c_{CL}(s) = s + a + ak_P \) is unstable iff \( k_P \leq -1 \)

- Discrete case \( c_{CL}(z) = z - e^{-ah} + (1 - e^{-ah})k_P \) is unstable iff \( k_P \leq -1 \) or \( k_P \geq \frac{1 + e^{-ah}}{1 - e^{-ah}} \)

\[ k_P = [1, 2, 3] \]
• The difference is in sampling + ZOH!
• ZOH brings in - roughly speaking - a time delay \( h/2 \)
• Compare

\[
P(s) = \frac{a}{s + a}
\]

\[
P_{\text{ZOH}}(s) = \frac{a}{s + a} e^{-sh/2}
\]

• What has finite GM!
• That's why it's better to count on it in a continuous design

Nyquist Diagram
System: csd
Gain Margin (dB): 11.6
At frequency (rad/s): 3.67
Closed loop stable? Yes
• Because a general controller (system) can be implemented by integrators

\[ C(s) = \frac{u(s)}{e(s)} = \frac{a_0 s^n + \cdots + a_1 s + a_0}{b_0 s^n + \cdots + b_1 s + b_0} \]

• We derive a discrete approximation for one (every) integrator

\[ C(s) = \frac{u(s)}{e(s)} = \frac{1}{s} \quad u(t) = u(0) + \int_0^t e(\tau) d\tau \]

• Output per a sampling period is

\[ u(kh + h) = u(kh) + \int_{kh}^{kh+h} e(\tau) d\tau \]

\[ \int_{kh}^{kh+h} e(\tau) d\tau = u(kh + h) - u(kh) \]

Different approaches approximate the integral by using values in discrete sampling moments.
Forward differentiation method

- Replacing the differential by forward differentiation we approximate the red area by green rectangle

\[ \int_{kh}^{kh+h} e(\tau) d\tau \approx he(kh) \]

\[ u(kh+h) = u(kh) + \int_{kh}^{kh+h} e(\tau) d\tau = u(kh) + he(kh) \]

- Using the z-transformation

\[ zu(z) = u(z) + he(z) \]

\[ \frac{u(z)}{e(z)} = \frac{h}{z-1} \]

\[ \frac{1}{s} \approx \frac{h}{z-1} \]

\[ s \approx \frac{z-1}{h} \]

- Thee method is also known as Euler Approximation
Replacing the differential by backward differentiation we approximate the red area by green rectangle

\[ \int_{kh}^{kh+h} e(\tau) d\tau \approx he(kh + h) \]

\[ u(kh + h) = u(kh) + \int_{kh}^{kh+h} e(\tau) d\tau \]

\[ = u(kh) + he(kh + h) \]

Using the z-transformation

\[ zu(z) = u(z) + zhe(z) \]

\[ u(z) = \frac{zh}{e(z)} \]

\[ \frac{1}{s} \approx \frac{zh}{z - 1} \]
Replacing the differential by bilinear differentiation we approximate the red area by green rectangle

\[ \int_{kh}^{kh+h} e(\tau)d\tau \approx \frac{h}{2}[e(hk) + e(hk + h)] \]

\[ u(kh + h) = u(kh) + \int_{kh}^{kh+h} e(\tau)d\tau \]

\[ = u(kh) + \frac{h}{2}[e(hk) + e(hk + h)] \]

Using the z-transformation

\[ zu(z) = u(z) + \frac{h}{2}e(z) + \frac{h}{2}ze(z) \]

\[ \frac{u(z)}{e(z)} = \frac{h}{2} \frac{z + 1}{z - 1} \]

\[ \frac{1}{s} \approx \frac{h}{2} \frac{z + 1}{z - 1} \]

\[ s \approx \frac{2}{h} \frac{z-1}{z+1} = \frac{2}{h} \frac{1-z^{-1}}{1+z^{-1}} \]
Attributes of approximations

Order
• All these transformations preserve the system order and the number of poles
  Higher order approximation are not used because the order would increase

Aliasing
• Beware of a stroboscopic effect: The controller incorrectly responds to the
  incorrectly sampled (alised) signal: error or reference
• “Anti-aliasing filter“ can help: The high frequency signals do not act
  incorrectly, they are invisible
• But the filter adds a phase delay and potentially destabilizes the CL

Stability OL
• If $C(s)$ is stable, is $C(z)$ stable? Comparison later. Minimal phase in details.

Stability CL:
• Although a continuous CL system is designed to be stable, with a discrete
  controller it might be unstable
• We need to calculate the discrete-time system with sampler and ZOH,
  connect the discrete controller to it and verify the discrete CL stability!
Stability of a cont. regulator and its approximation (OL)

Forward differentiation

\[ z = e^{sh} \approx 1 + sh \]
\[ \text{Re}(s) < 0 \rightarrow \text{Re}(s) < 1 \]

\[ s = \frac{z - 1}{h} = \frac{z}{h} - \frac{1}{h} \]
\[ \text{Re}|z| < 1 \rightarrow \text{Re}|1 + sh| < 1 \]

• Stable continuous controller with high frequency or low damped modes (poles) has unstable discrete approximation
Backward differentiation

\[ z = \frac{1}{1 - sh} \]

- Preserves stability
- Although it has continuous low-damped modes, discrete has not.

\[ s = \frac{1}{h} - \frac{1}{hz} \]

Tustin’s method

\[ z = \frac{1 + sh/2}{1 - sh/2}, \quad s = \frac{2}{h} \frac{z - 1}{z + 1} \]

- It preserves stability (and minimum phase)
- The stable area transformation is one-to-one, therefore it is used most often
Tustin’s method: Numerical example

• Manually

\[ C(s) = \frac{a}{a + s} \quad \rightarrow \quad C_{\text{Tustin}}(z) = \frac{a}{a + \frac{2}{h} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right)} = \frac{(1 + z^{-1}) ah}{ah + 2(1 - z^{-1})} \]

\[ s = \frac{2}{h} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) \]

• In Matlab – CSTbx:

```matlab
>> a=2; h=4; C=a./(a+s)
C = 2
     ----- 
     2 + s

>> CTustin=c2d(tf(C),h,'tustin')
Transfer function:
0.8 z + 0.8
-------------
    z + 0.6
Sampling time: 4
```
Continuous controller with a transfer function

\[ C(s) = \frac{a}{a + s} \]

- Approximation by forward differentiation

\[ C_{\text{forward}}(z) = \frac{a}{a + \frac{z-1}{h}} = \frac{ah}{z + ah - 1} \]

- Backward differentiation

\[ C_{\text{backward}}(z) = \frac{a}{a + \frac{z-1}{zh}} = \frac{ahz}{z(ah+1) - 1} \]

- And Tustin’s method

\[ C_{\text{Tustin}}(z) = \frac{a}{a + \frac{2}{h} \left( \frac{z-1}{z+1} \right)} = \frac{ah(z+1)}{(ah+2)z + ah - 2} \]
Continuous regulator:

\[ C(s) = \frac{s + 1}{(0.1s + 1)(0.01 + s)} \]

Approximate:
- Forward differentiation
- Backward differentiation
- Tustin’s method

\[
\begin{align*}
S &= (z-1)/h; \\
Cpd &= (S+1)/((0.1*S+1)*(0.01*S+1)), props(Cpd,h); \\
Czd &= (S+1)/((0.1*S+1)*(0.01*S+1)), props(Czd,h); \\
Ctu &= c2d(tf(C),h,'tustin')
\end{align*}
\]

Transfer function:

\[
\frac{5.857 z^2 + 0.2857 z - 5.571}{z^2 - 0.1714 z - 0.2571}
\]

\[ \frac{\omega_s}{2} = \frac{\omega_N}{2} = 63 \text{ rad/s} \]

Tustin’s approximation: the best, up to the Nyquist frequency!
Exampe: Prewarping

\[ bode(C, Ctustin, Cpw) \]

\[ \omega_{pw} = 12.8 \text{ rad/s} \]
MPZ (Matched pole-zero)

• It is based on pole/zero relationship \( z_i = e^{s_i h} \) for continuous and sampled signal.
• If it is possible we add zero in \( z^{-1} = -1 \), therefore \( (z^{-1} + 1) \) leading to averaging current and previous values.
• The method is simple and practical, although not very substantiated.

MPZ Procedure

1. Calculate the zeros and poles of the continuous controller \( C(s) \)
2. Set \( C(z) \) so that \( z_i = e^{s_i h} \)
3. If it is possible, add the numerator members \((z + 1)\) so that numerator degree = denominator degree
4. Set low frequency gain of \( C(z) \) same as it was in \( C(s) \)
MPZ Method

MPZ for

\[ C(s) = K_C \frac{s + a}{s + b} \]

\[ C_{MPZ}(z) = K_D \frac{z - e^{-ah}}{z - e^{-bh}} \]

\[ z_i = e^{s_i h} \]

\[ C(0) = K_C \frac{a}{b} = C_{MPZ}(1) = K_D \frac{1 - e^{-ah}}{1 - e^{-bh}} \]

\[ K_D = K_C \frac{a}{b} \frac{1 - e^{-bh}}{1 - e^{-ah}} \]

MPZ for

\[ C(s) = K_C \frac{s + a}{s(s + b)} \rightarrow C_{MPZ,1}(z) = K_D \frac{z - e^{-ah}}{(z - 1)(z - e^{-bh})} \rightarrow C_{MPZ}(z) = K_D \frac{(z + 1)(z - e^{-ah})}{(z - 1)(z - e^{-bh})} \]

\[ K_D = K_C \frac{a}{2b} \frac{1 - e^{-bh}}{1 - e^{-ah}} \]
Example: ZOH

For continuous transfer function

\[ G(s) = \frac{a}{s(s+a)} \]

Discrete TF is

\[ G(z) = \left(1 - z^{-1}\right) \frac{a}{s(s+a)} = \left(1 - z^{-1}\right) \frac{(1-e^{-ah})z^{-1}}{(1-z^{-1})(1-e^{-ah}z^{-1})} \]

\[ \alpha = e^{-ah} \]

\[ \frac{1 - \alpha}{z - \alpha} \]

\[
-1 \left\{ \frac{a}{s(s+a)} \right\} = -1 \left\{ \frac{1}{s} - \frac{1}{s+a} \right\} = 1 - e^{-at} \Rightarrow 1 - e^{-ah}
\]

\[
\left\{1 - e^{-ah}\right\} = \frac{z}{z-1} - \frac{z}{z-e^{-ah}} = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-ah}z^{-1}} = \frac{(1-e^{-ah})z^{-1}}{(1-z^{-1})(1-e^{-ah}z^{-1})}
\]

\[
\gg \text{sdf(c2d(tf(1/(s+1)),1,'zoh'))}
\]

\[
\text{ans} = \begin{array}{c} 0.6321 \end{array}
\]

\[
\begin{array}{c}
\text{reduced} \\
(z-0.3679)
\end{array}
\]
Example: Time delay

- Continuous TF of the mixer part is

\[ G(s) = \frac{T_e(s)}{T_{ec}(s)} = e^{\tau_d s} F(s) = e^{\tau_d s} \frac{s}{s + a} \]

- For values \( a = 1, h = 1, \tau_d = 1.5 \) We find discrete TF

- Because time delay \( \tau_d \) is not integer multiple of sampling period \( h \), It is devided to

\( \tau_d = lh - mh, l \in Z, m < 1, m \in R \quad 1.5 = 2 - 0.5, h = 1, l = 2, \lambda = 0.5 \)

- So we get

\[ \frac{G(s)}{s} = e^{-lhs} e^{-mhs} \frac{F(s)}{s} \]

- And the element \( e^{-lhs} \) will change to \( Z^{-l} \)
Example: Time delay

- After substitution and partial fractions decomposition, we will get

\[ G(z) = (1 - z^{-1}) \left\{ \frac{G(s)}{s} \right\} = (1 - z^{-1}) z^{-l} \left\{ \frac{e^{-mhs}}{s} - \frac{e^{-mhs}}{s + a} \right\} \]

\[ m_h \text{ second shifted unit step and exponential with the same time shift} \]

- Time shift is less than the entire period \((m < 1)\),
  It does not take a sample for \(t < 0\)

- Samples are

\[ 1(kh) \rightarrow z / (z - 1) \]

\[ e^{-ah(k+m)} 1(kh) \rightarrow z e^{-amh} / (z - e^{-ah}) \]

- Therefore

\[ G(z) = \frac{z - 1}{z} \frac{1}{z^l} \left( \frac{z}{z - 1} - \frac{z e^{-amh}}{z - e^{-ah}} \right) = (1 - e^{-amh}) \frac{z + \alpha}{z^l (z - e^{-ah})} \]
Example: Time delay

- For values \( a = 1, h = 1, \tau_d = 1.5 \)

\[
\begin{align*}
\text{>> } G &= \text{tf}([1],[1 1], 'iodelay', 1.5) \\
\text{Transfer function:} \\
&= \frac{\exp(-1.5s)}{s + 1}
\end{align*}
\]

\[
\begin{align*}
\text{>> } G_d &= \text{c2d}(G, 1, 'zoh') \\
\text{Transfer function:} \\
&= \frac{0.3935 z + 0.2387}{z^2 - 0.3679 z}
\end{align*}
\]

Sampling time: 1

\[
\begin{align*}
\text{>> } G_d &= \text{/.3935} \\
\text{Transfer function:} \\
&= \frac{z + 0.6065}{z^2 - 0.3679 z}
\end{align*}
\]

Sampling time: 1

- Continuous TF doesn’t have a zero

- Discrete TF has a zero in

\[
\alpha = -\frac{e^{-amh} - e^{-ah}}{1 - e^{-amh}}
\]

\[
\begin{align*}
\text{>> } \alpha &= (\exp(-m) - \exp(-1))./(1 - \exp(-m)); \text{ plot}(m, -\alpha) \\
\text{>> } \text{syms } m; m_{sb} &= \text{solve}('(\exp(-m) - \exp(-1))/(1 - \exp(-m)) = 1') \\
m_{sb} &= \text{log}(1/2*\exp(-1)+1/2) \\
\text{>> } \text{vpa}(m_{sb}, 3) \quad \text{ans} = .380
\end{align*}
\]
• For continues TF we calculate

\[
\frac{G(s)}{s^2} = \frac{1}{s^2}
\]

\[
\left\{ \frac{G(s)}{s^2} \right\} = \left\{ \frac{1}{s^4} \right\} = \frac{t^3}{3!} \rightarrow \frac{(kh)^3}{6} \rightarrow \left\{ \frac{1}{6} (kh)^3 \right\} = \frac{h^3}{6} \left\{ k^3 \right\} = \frac{h^3}{6} \frac{z^2 + 4z + 1}{(z - 1)^4}
\]

• Then discrete TF is

\[
G(z) = \frac{(z - 1)^2}{hz} \frac{h^3}{6} \frac{z(z^2 + 4z + 1)}{(z - 1)^4}
\]

\[
= \frac{h^2}{6} \frac{(z^2 + 4z + 1)}{(z - 1)^2}
\]

\[
>> Gz=c2d(tf(1/s^2),1,'foh')
\]

Transfer function:
0.1667 z^2 + 0.6667 z + 0.1667
--------------------------------------
z^2 - 2 z + 1

Sampling time: 1

\[
>> Gzp=sdf(Gz)
\]

\[
Gzp =
0.1667(z+3.7321)(z+0.2679)
--------------------------------------
(z-1)(z-1)
\]
```matlab
>> D=5/(s+5); DD=zpk(D); T=1/15;
>> DDtustin=c2d(DD,T,'tustin'),
    DDmpz=c2d(DD,T,'matched'), ...
    DDzoh=c2d(DD,T,'zoh'), DDfoh=c2d(DD,T,'foh'), ...
Zero/pole/gain:
    0.14286 (z+1)
    --------------          Sampling time: 0.066667
    (z-0.7143)
Zero/pole/gain:
    0.28347
    ---------          Sampling time: 0.066667
    (z-0.7165)
Zero/pole/gain:
    0.28347
    ---------          Sampling time: 0.066667
    (z-0.7165)
Zero/pole/gain:
    0.14959 (z+0.8949)
    ------------------          Sampling time: 0.066667
    (z-0.7165)
>> bode(DD,DDtustin,DDmpz,DDzoh,DDfoh)
>> omegas=2*pi/T                           omegas = 94.2478
```
\[ G(s) = \frac{5}{s + 5} \]

\[ h = 1/15 \text{ s} \]

\[ \omega_s \approx 94 \text{ rad} \]

- All OK to

\[ \omega_N / 2 = \omega_s / 4 \]
Discretization of the state model: Derivation

Discrete state model + 0. order shaping part

• We come out of the continuous model

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

• If the system in time \( t_0 \) has a state \( x(t_0) \) then for \( t \geq t_0 \) the state is:

\[ x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \]

• We need to know the input over the interval \([t_0, t)\)
Discretization of the state model: Derivation

- We are interested in relation between state in time $t_{k+1}$ and state in time $t_k$ considering ZOH i.e. constant input
- $u_k = u(\tau), \tau \in [t_k, t_{k+1})$ During the sampling interval
- consider $h = t_{k+1} - t_k$ and we get

$$
\begin{align*}
\mathbf{x}(t_{k+1}) &= e^{A(t_{k+1}-t_k)}\mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}\mathbf{B}u(\tau)\,d\tau \\
&= e^{Ah}\mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}\,d\tau \mathbf{B}u(t_k) \\
&= e^{Ah}\mathbf{x}(t_k) + \left(\int_0^h e^{Av}\,dv\right) \mathbf{B}u(t_k)
\end{align*}
$$

- Sampling does not change the output equation so we get

$$
\mathbf{x}(t_{k+1}) = \mathbf{F}\mathbf{x}(t_k) + \mathbf{G}u(t_k) \\
y(t_k) = \mathbf{C}\mathbf{x}(t_k) + Du(t_k)
$$

$\mathbf{F} = e^{Ah}$

$\mathbf{G} = \left(\int_0^h e^{Av}\,dv\right)\mathbf{B}$
There are many methods for matrix exp. calculation:

\[ F = e^{Ah}, \quad G = \left( \int_0^h e^{Av} \, dv \right) B \]

- Decomposition in the Taylor series

\[
V = \int_0^h e^{Av} \, dv = I + h A + \frac{Ah^2}{2!} + \frac{Ah^3}{3!} + \ldots + \frac{Ah^{i+1}}{(i+1)!} + \ldots
\]

\[
e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.
\]

- Jordan form (eigenvalues)

\[
A = V \text{ diag}\{\lambda_i\} V^{-1} \quad \Rightarrow \quad e^{Ah} = V \text{ diag}\{e^{h\lambda_i}\} V^{-1}
\]

- Cayley-Hamilton theorem

- Matlab function `expm` – Pade approximation
• Continues 1st order system
\[ \dot{x} = \alpha x + \beta u \]

• With sampling period \( h \) is
\[ F = e^{Ah} = e^{\alpha h} \]
\[ G = \left( \int_0^h e^{\alpha v} \, dv \right) \beta = \frac{\beta}{\alpha} \left( e^{\alpha v} - 1 \right) \]

• So the discrete system with ZOH is
\[ x(k + 1) = e^{\alpha h} x(k) + \frac{\beta}{\alpha} \left( e^{\alpha v} - 1 \right) u(k) \]
Example

- For double integrator
  \[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \]

- With sampling period \( h \) is
  \[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \]

\[
F = e^{Ah} = I + Ah + A^2 h^2/2 + \ldots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} + 0 + \ldots = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}
\]

\[
G = \int_0^h e^{Av} B \, dv = \int_0^h \begin{bmatrix} v \\ 1 \end{bmatrix} \, dv = \begin{bmatrix} h^2/2 \\ h \end{bmatrix}
\]

- So the discrete system with ZOH is
  \[
x(k+1) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} h^2/2 \\ h \end{bmatrix} u(k)
  \]
  \[ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \]
• In previous examples: numerator degree in $z = \text{denumerator degree in } z$
• Therefore differential controller equation
  \[ u(k) + \text{parts with } k-1, \ldots = ce(k) + \text{parts with } k-1, k-2, \ldots \]
• Such a digital controller must compute immediately i.e., the delay resulting from the non-zero calculation time is neglected.
• This is practically acceptable only if the calculation time is $< 1$. Controller has at least one step delay
• numerator degree in $z < \text{denumerator degree in } z$
• We can get it by MPZ incompleteness
• Or we have to „add“ delay to system
• This is not a time delay (only way of indexing)