Exercises for lectures
21 – Discrete-time equivalents

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Introduction: CL stability for cont. and disc. control

Designing emulations: CL stability of cont. control does not guarantees CL stability of disc. control! We need to test "discrete stability"!

We will show it on the P regulator – simple one. We do not "approximate“ it, but use "as is" for discrete control.

- For system $P(s)=\dfrac{a}{s+a}, a>0$, regulator $C(s)=k_p$ for cont. control

  CL characteristic polynomial is $c_{CL}(s)=s+a+ak_P$

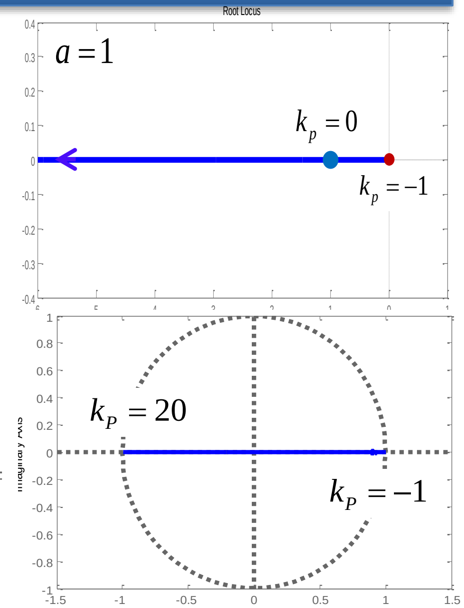
- Now, use the same controller in discrete control $C(z)=C(s)=k_p$
- For the discrete case analysis we use a "discrete system model“ (continuous sys. + sampler + ZOH shaper)

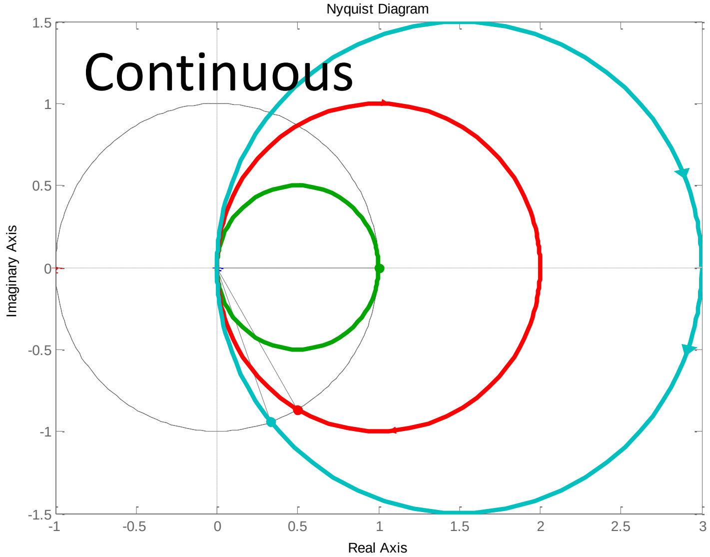
$$P(z)=\frac{1-e^{-ah}}{z-e^{-ah}}$$

- Resulting CL characteristic polynomial is $c_{CL}(z)=z-e^{-ah}+(1-e^{-ah})k_P$

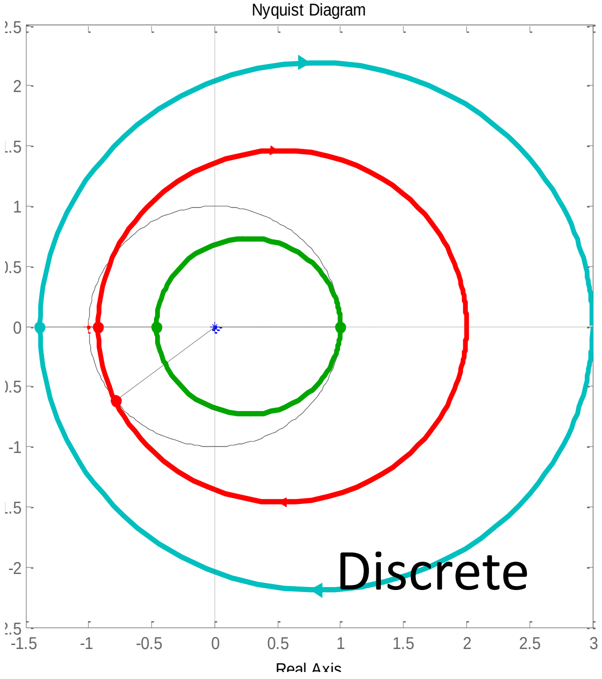
Introduction: CL stability for cont. and disc. control

- Continuous case $c_{CL}(s) = s + a + ak_P$ is unstable iff $k_P \le -1$
- Discrete case $c_{CL}(z) = z - e^{-ah} + (1 - e^{-ah})k_P$ is unstable iff $k_P \le -1$ or $k_P \ge \dfrac{1+e^{-ah}}{1-e^{-ah}}$





$k_P = [1, 2, 3]$



Continuous case has infinite GM

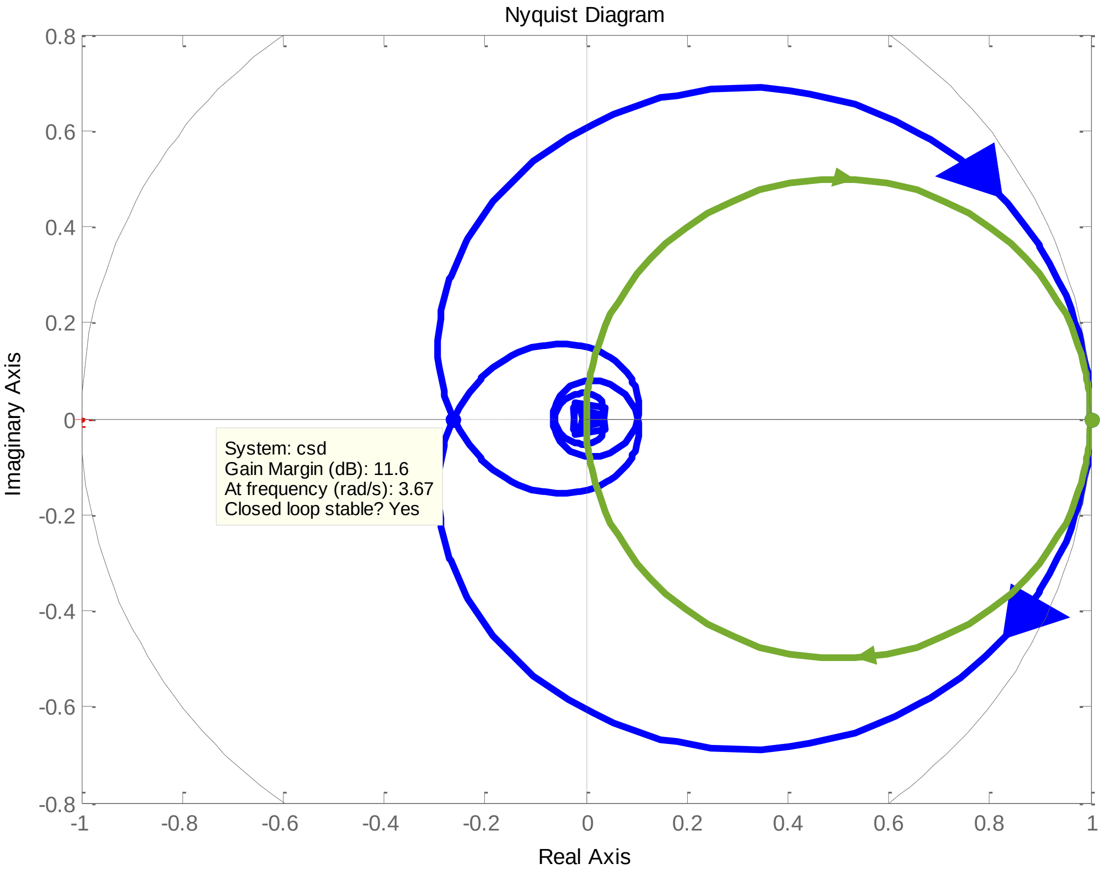
Discrete case has finite GM

CL stability for cont. and disc. control

- The difference is in sampling + ZOH !
- ZOH brings in - roughly speaking - a time delay $h/2$
- Compare

$$P(s) = \frac{a}{s+a}$$

$$P_{ZOH}(s) = \frac{a}{s+a} e^{-sh/2}$$

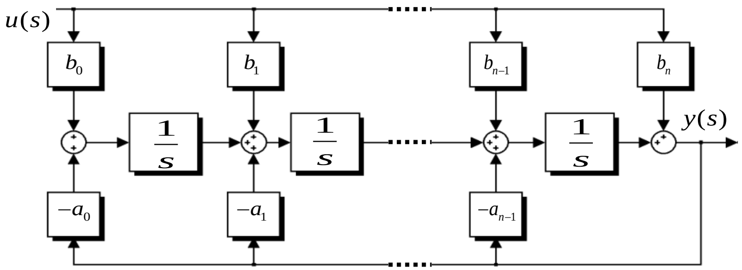


- What has finite GM!
- That's why it's better to count on it in a continuous design

Derivation of the Approximation method

- Because a general controller (system) can be implemented by integrators

$$C(s)=\frac{u(s)}{e(s)}=\frac{a_n s^n+\cdots+a_1 s+a_0}{b_n s^n+\cdots+b_1 s+b_0}$$



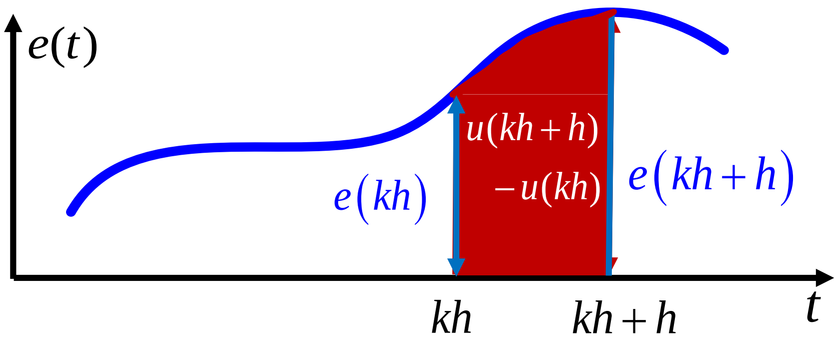
- We derive a discrete approximation for one (every) integrator

$$C(s)=\frac{u(s)}{e(s)}=\frac{1}{s} \qquad u(t)=u(0)+\int_0^t e(\tau)d\tau$$

- Output per a sampling period is

$$u(kh+h)=u(kh)+\int_{kh}^{kh+h} e(\tau)d\tau$$

$$\int_{kh}^{kh+h} e(\tau)d\tau=u(kh+h)-u(kh)$$

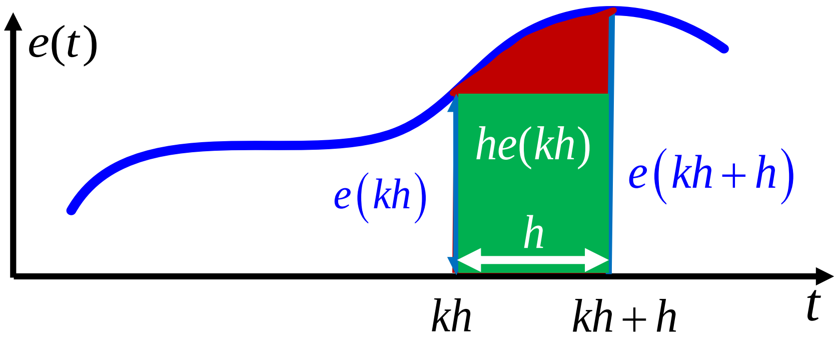


Different approaches approximate the integral by using values in discrete sampling moments.

Forward differentiation method

- Replacing the differential by forward differentiation we approximate the red area by green rectangle

$$\int_{kh}^{kh+h} e(\tau)d\tau \approx he(kh)$$



$$u(kh+h) = u(kh) + \int_{kh}^{kh+h} e(\tau)d\tau$$

$$= u(kh) + he(kh)$$

- Using the *z*-transformation

$$zu(z) = u(z) + he(z)$$

$$\frac{u(z)}{e(z)} = \frac{h}{z-1} \quad \Rightarrow \quad \frac{1}{s} \cong \frac{h}{z-1} \quad \Rightarrow \quad \boxed{s \cong \frac{z-1}{h}}$$

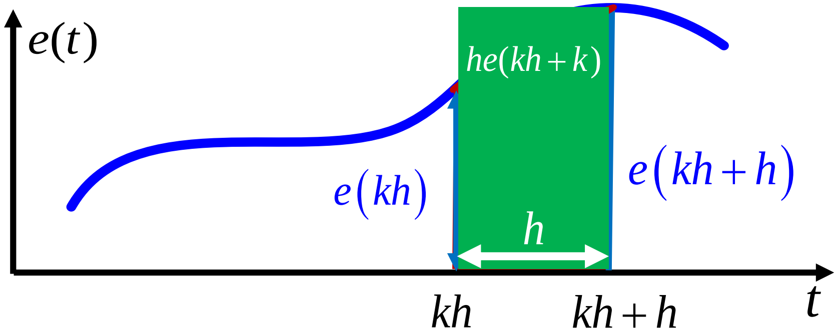
- Thee method is also known as Euler Approximation

Backward differentiation method

- Replacing the differential by backward differentiation we approximate the red area by green rectangle

$$\int_{kh}^{kh+h} e(\tau)d\tau \approx he(kh+h)$$

$$u(kh+h) = u(kh) + \int_{kh}^{kh+h} e(\tau)d\tau$$
$$= u(kh) + he(kh+h)$$



- Using the z-transformation

$$zu(z) = u(z) + zhe(z)$$
$$\frac{u(z)}{e(z)} = \frac{zh}{z-1}$$

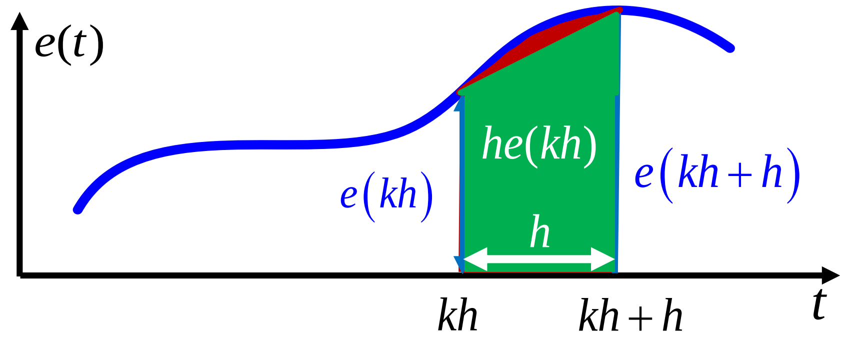
$$\frac{1}{s} \cong \frac{zh}{z-1}$$

$$s \cong \frac{z-1}{zh}$$

Tustin's (bilinear) method

- Replacing the differential by bilinear differentiation we approximate the red area by green rectangle

$$\int_{kh}^{kh+h} e(\tau)d\tau \approx \frac{h}{2}[e(hk)+e(hk+h)]$$



$$u(kh+h) = u(kh) + \int_{kh}^{kh+h} e(\tau)d\tau$$

$$= u(kh) + \frac{h}{2}[e(hk)+e(hk+h)]$$

The content of the green trapezoid

$$\frac{h}{2}[e(hk)+e(hk+h)]$$

- Using the *z*-transformation

$$zu(z) = u(z) + \frac{h}{2}e(z) + \frac{h}{2}ze(z)$$

$$\frac{u(z)}{e(z)} = \frac{h}{2}\frac{z+1}{z-1} \quad \Rightarrow \quad \frac{1}{s} \cong \frac{h}{2}\frac{z+1}{z-1} \quad \Rightarrow \quad s \cong \frac{2}{h}\frac{z-1}{z+1} = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}$$

Attributes of approximations

Order

- All these transformations preserve the system order and the number of poles Higher order approximation are not used because the order would increase

Aliasing

- Beware of a stroboscopic effect: The controller incorrectly responds to the incorrectly sampled (alised) signal: error or reference
- “Anti-aliasing filter“ can help: The high frequency signals do not act incorrectly, they are invisible
- But the filter adds a phase delay and potentially destabilizes the CL

Stability OL

- If C(s) is stable, is C(z) stable? Comparison later. Minimal phase in details.

Stability CL:

- Although a continuous CL system is designed to be stable, with a discrete controller it might be unstable
- We need to calculate the discrete-time system with sampler and ZOH, connect the discrete controller to it and verify the discrete CL stability!

Stability of a cont. regulator and its approximation (OL)

Forward differentiation

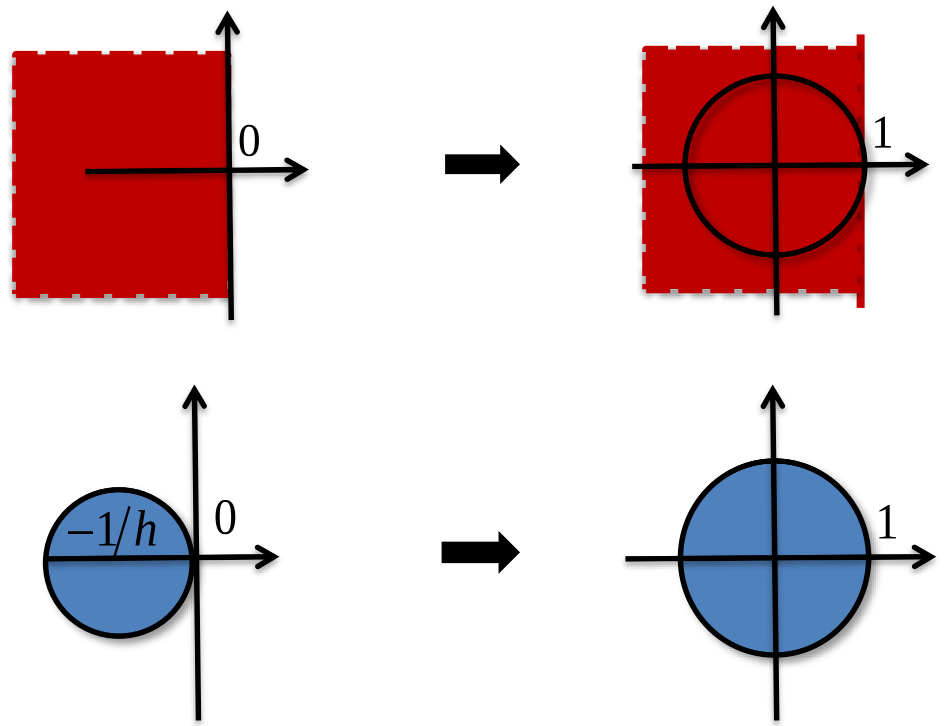
$$z = e^{sh} \approx 1 + sh$$

$$\operatorname{Re} s < 0 \rightarrow \operatorname{Re} s < 1$$

$$s = \frac{z-1}{h} = \frac{z}{h} - \frac{1}{h}$$

$$\operatorname{Re}|z| < 1 \rightarrow \operatorname{Re}|1 + sh| < 1$$

continuous | discrete

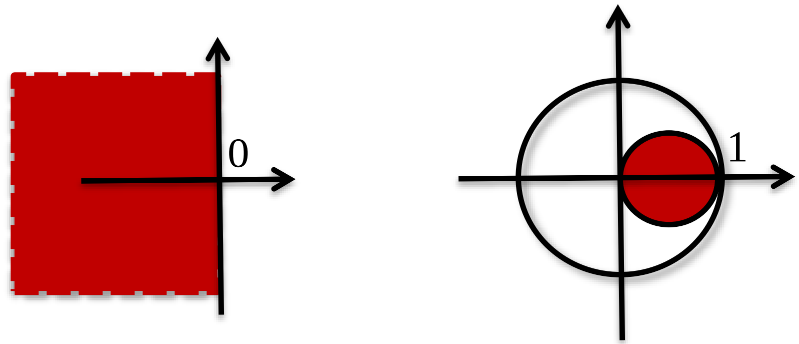


- Stable continuous controller with high frequency or low damped modes (poles) has unstable discrete approximation

Stability of a cont. regulator and its approximation (OL)

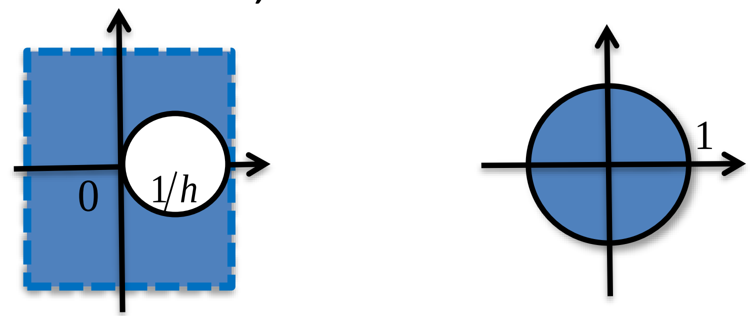
Backward differentiation

$$z = \frac{1}{1 - sh}$$



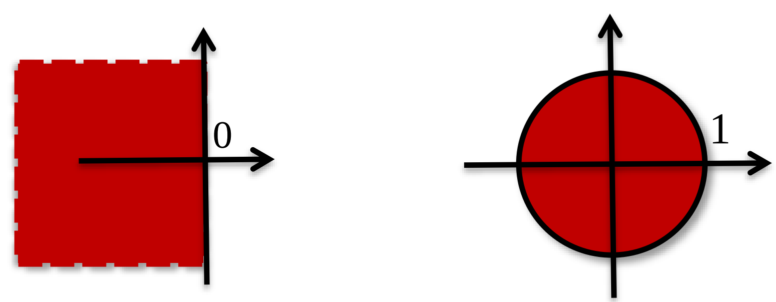
- Preserves stability
- Although it has continuous low-damped modes, discrete has not.

$$s = \frac{1}{h} - \frac{1}{hz}$$



Tustin's method

$$z = \frac{1 + sh/2}{1 - sh/2}, s = \frac{2}{h}\frac{z-1}{z+1}$$



- It preserves stability (and minimum phase)
- The stable area transformation is one-to-one, therefore it is used most often

Tustin's method: Numerical example

- Manually

$$C(s) = \frac{a}{a+s} \quad \Rightarrow \quad C_{\text{Tustin}}(z) = \frac{a}{a + \frac{2}{h}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{\left(1+z^{-1}\right)ah}{ah + 2\left(1-z^{-1}\right)}$$

$$s = \frac{2}{h}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

- In Matlab – CSTbx:

```
>> a=2;h=4;C=a./(a+s)
C =    2
     -----
     2 + s

>> CTustin=c2d(tf(C),h,'tustin')
Transfer function:
0.8 z + 0.8
-----------
  z + 0.6
Sampling time: 4
```

Examples

Continuous controller with a transfer function $C(s)=\dfrac{a}{a+s}$

- Approximation by forward differentiation

$$C_{\text{forward}}(z)=\frac{a}{a+\dfrac{z-1}{h}}=\frac{ah}{z+ah-1}$$

- Backward differentiation

$$C_{\text{backward}}(z)=\frac{a}{a+\dfrac{z-1}{zh}}=\frac{ahz}{z(ah+1)-1}$$

- And Tustin's method

$$C_{\text{Tustin}}(z)=\frac{a}{a+\dfrac{2}{h}\left(\dfrac{z-1}{z+1}\right)}=\frac{ah\left(z+1\right)}{\left(ah+2\right)z+ah-2}$$

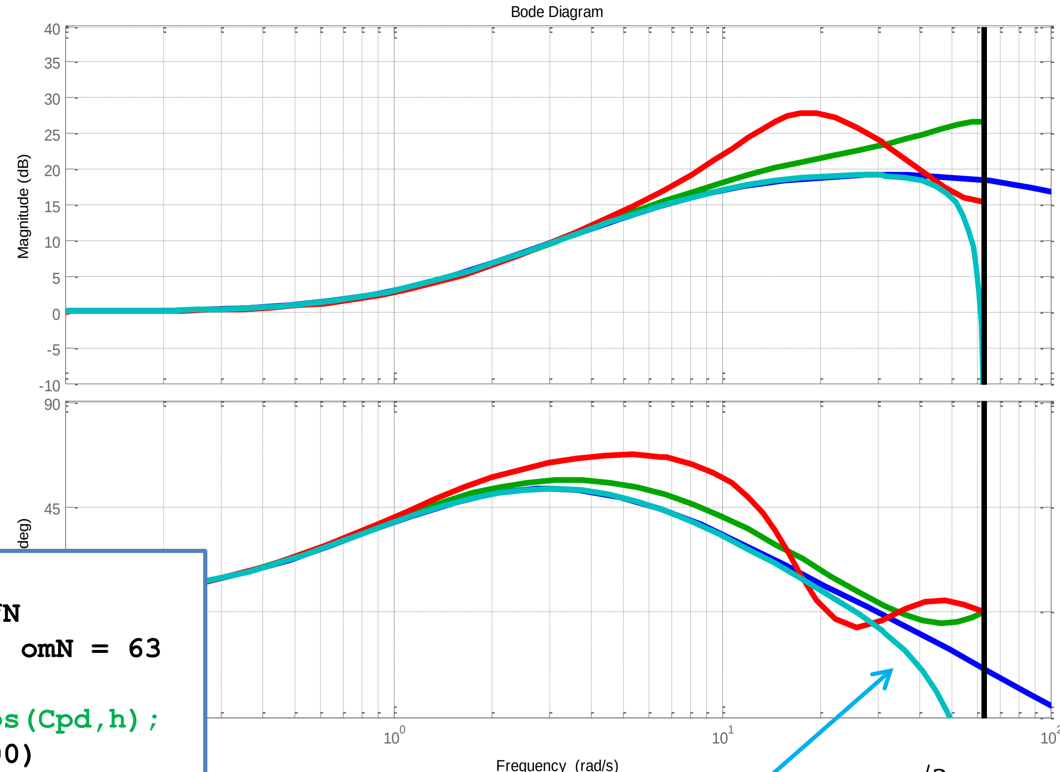
Examples

Continuous regulator:

$$C(s)=\frac{s+1}{(0.1s+1)(0.01+s)}$$

Approximate:

- Forward differentiation
- Backward differentiation
- Tustin's method



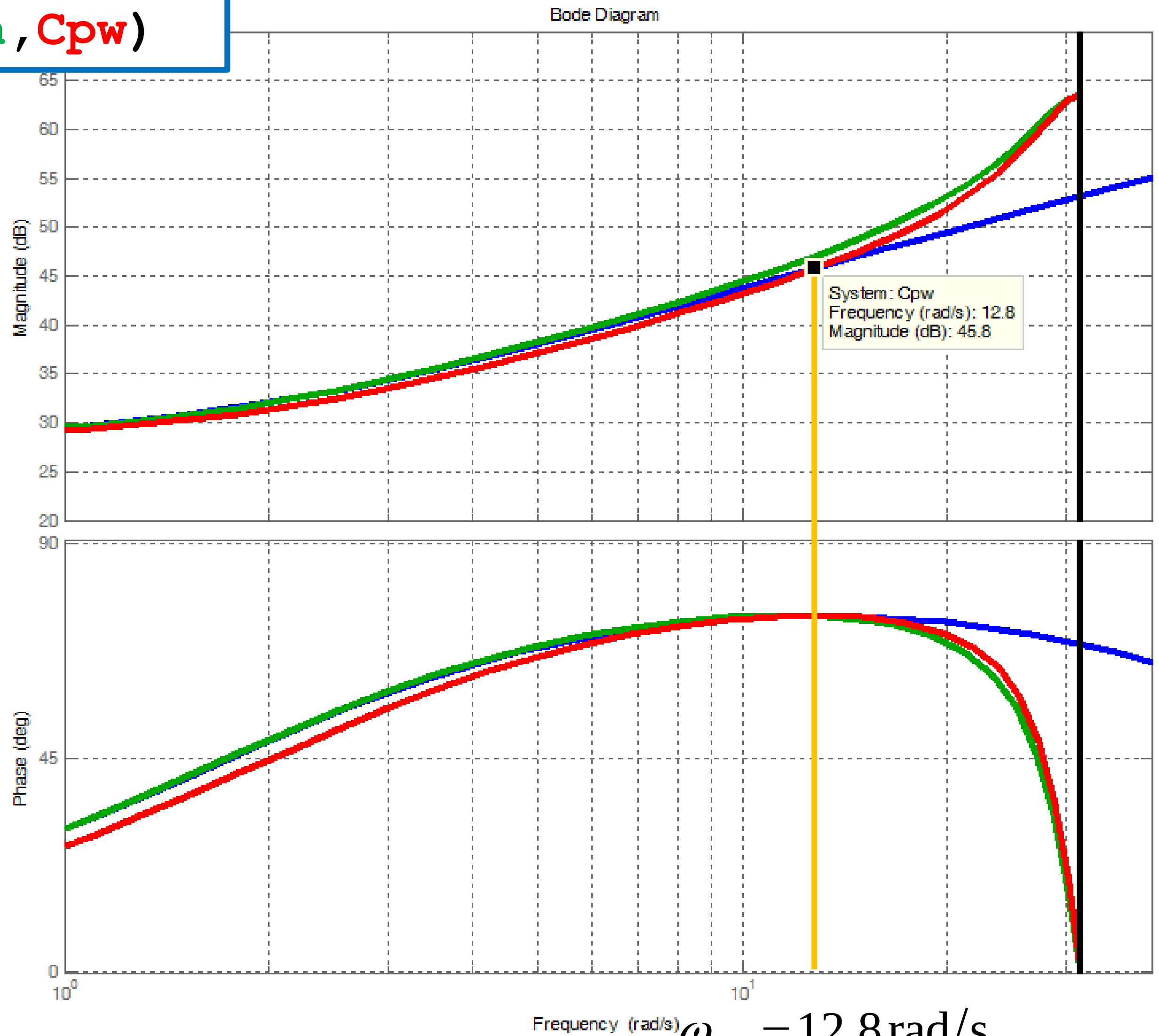

```
>> C=(s+1)/((0.1*s+1)*(0.01*s+1));h=.05;
>> fs=1/h,fN=fs/2,oms=2*pi*fs,omN=2*pi*fN
h = 0.0500, fs = 20, fN = 10, oms = 12^, omN = 63
>> S=(z-1)/h;
>> Cpd=(S+1)/((0.1*S+1)*(0.01*S+1)),props(Cpd,h);
Cpd =  50(z-0.9500) / (z+4.0000)(z-0.5000)
>> S=(z-1)./h*z;
>> Czd=(S+1)/((0.1*S+1)*(0.01*S+1)),props(Czd,h);
>> Ctu=c2d(tf(C),h,'tustin')
Transfer function:
5.857 z^2 + 0.2857 z - 5.571
----------------------------
  z^2 - 0.1714 z - 0.2571
>> bode(tf(C),tf(Cpd),tf(Czd),Ctu)
```

$\omega_S/2=$
$\omega_N=63\,\mathrm{rad/s}$

Tustin's approximation: the best, up to the Nyquist frequency!

Exampe: Prewarping


```
>> bode(C,Ctustin,Cpw)
```



$$\omega_{pw} = 12.8\,\mathrm{rad/s}$$

MPZ Method

MPZ (Matched pole-zero)

- It is based on pole/zero relationship $z_i = e^{s_i h}$ for continuous and sampled signal.
- If it is possible we add zero in $z^{-1} = -1$, therefore $(z^{-1} + 1)$ Leading to averaging current and previous values
- The method is simple and practical, although not very substantiated

MPZ Procedure

1. Calculate the zeros and poles of the continuous controller $C(s)$
2. set $C(z)$ so that $z_i = e^{s_i h}$
3. If it is possible, add the numerator members $(z+1)$ so that numerator degree = denominator degree
4. Set low frequenci gain of $C(z)$ same as it was in $C(s)$

MPZ Method

MPZ for

$$C(s) = K_C \frac{s+a}{s+b}$$

$$C_{MPZ}(z) = K_D \frac{z-e^{-ah}}{z-e^{-bh}}$$

$$z_i = e^{s_i h}$$

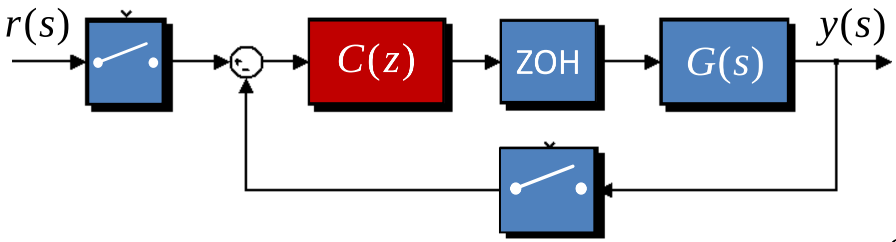
$$C(0) = K_C \frac{a}{b} \overset{!}{=} C_{MPZ}(1) = K_D \frac{1-e^{-ah}}{1-e^{-bh}} \Rightarrow K_D = K_C \frac{a}{b} \frac{1-e^{-bh}}{1-e^{-ah}}$$

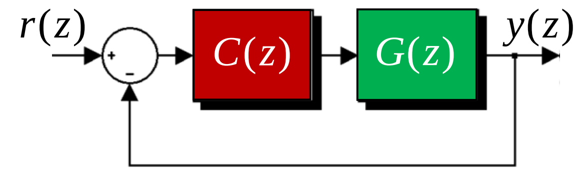
MPZ for

$$C(s) = K_C \frac{s+a}{s(s+b)} \rightarrow C_{MPZ,1}(z) = K_D \frac{z-e^{-ah}}{(z-1)(z-e^{-bh})} \rightarrow C_{MPZ}(z) = K_D \frac{(z+1)(z-e^{-ah})}{(z-1)(z-e^{-bh})}$$

$$K_D = K_C \frac{a}{2b} \frac{1-e^{-bh}}{1-e^{-ah}}$$

Example: ZOH





For continuous transfer function

$$G(s)=\frac{a}{s+a}$$

Discrete TF is

$$\mathcal{L}^{-1}\left\{\frac{a}{s(s+a)}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+a}\right\}=1-e^{-at}\ \rightarrow\ 1-e^{-akh}$$

$$\mathcal{Z}\left\{1-e^{-akh}\right\}=\frac{z}{z-1}-\frac{z}{z-e^{-ah}}$$

$$=\frac{1}{1-z^{-1}}-\frac{1}{1-e^{-ah}z^{-1}}$$

$$=\frac{(1-e^{-ah})z^{-1}}{(1-z^{-1})(1-e^{-ah}z^{-1})}$$

$$G(z)=(1-z^{-1})\mathcal{Z}\left\{\frac{a}{s(s+a)}\right\}=$$

$$=(1-z^{-1})\frac{(1-e^{-ah})z^{-1}}{(1-z^{-1})(1-e^{-ah}z^{-1})}$$

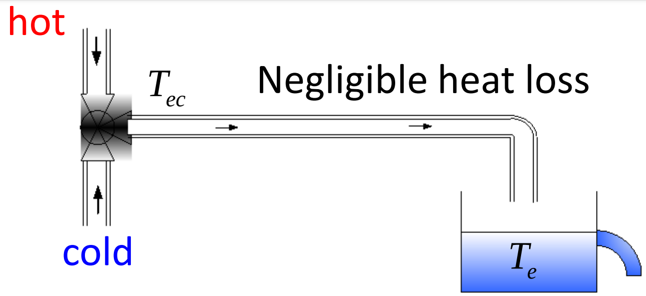
$$=\frac{1-\alpha}{z-\alpha},\qquad \alpha=e^{-ah}$$

```
>> sdf(c2d(tf(1/(s+1)),1,'zoh'))
ans =   0.6321
      ----------      reduced
       (z-0.3679)
```

Example: Time delay

- Continuous TF of the mixer part is

$$G(s)=\frac{T_e(s)}{T_{ec}(s)}=e^{\tau_d s}F(s)=e^{\tau_d s}\frac{s}{s+a}$$



- For values $a=1, h=1, \tau_d=1.5$ We find discrete TF
- Because time delay τ_d is not integer multiple of sampling period *h*, It is devided to

$$\tau_d = lh - mh, l\in Z, m<1, m\in R \quad \Rightarrow \quad 1.5=2-0.5, h=1, l=2, \lambda=0.5$$

- So we get

$$\frac{G(s)}{s}=e^{-lhs}\frac{e^{-mhs}F(s)}{s}$$

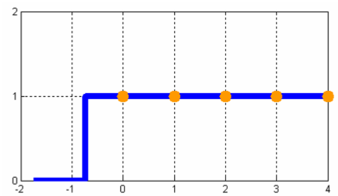
- And the element e^{-lhs} will change to z^{-l}

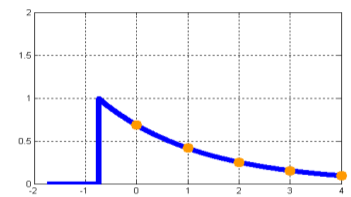
Example: Time delay

- After substitution and partial fractions decomposition, we will get

$$G(z)=\left(1-z^{-1}\right)\mathrm{Z}\left\{\frac{G(s)}{s}\right\}=\left(1-z^{-1}\right)z^{-l}\mathrm{Z}\left\{\frac{e^{-mhs}}{s}-\frac{e^{-mhs}}{s+a}\right\}$$

mh second shifted unit step and exponential with the same time shift





- Time shift is less than the entire period ($m<1$),
  It does not take a sample for $t<0$
- Samples are

$$1(kh) \Rightarrow z/(z-1)$$

$$e^{-ah(k+m)}1(kh) \Rightarrow ze^{-amh}/(z-e^{-ah})$$

- Therefore

$$G(z)=\frac{z-1}{z}\frac{1}{z^{l}}\left(\frac{z}{z-1}-\frac{ze^{-amh}}{z-e^{-ah}}\right)=\left(1-e^{-amh}\right)\frac{z+\alpha}{z^{l}\left(z-e^{-ah}\right)}$$

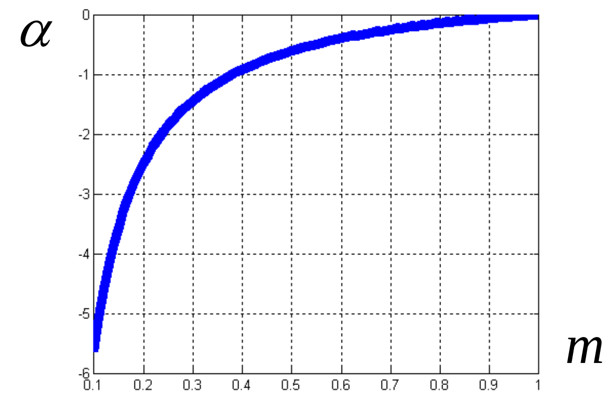
Example: Time delay

- For values $a=1, h=1, \tau_d=1.5$

```
>> G=tf([1],[1 1],'iodelay',1.5)
Transfer function:
                1
exp(-1.5*s) * -----
              s + 1
>> Gd=c2d(G,1,'zoh')
Transfer function:
         0.3935 z + 0.2387
z^(-1) * -----------------
          z^2 - 0.3679 z
Sampling time: 1
>> Gd/.3935
Transfer function:
           z + 0.6065
z^(-1) * --------------
         z^2 - 0.3679 z
Sampling time: 1
```

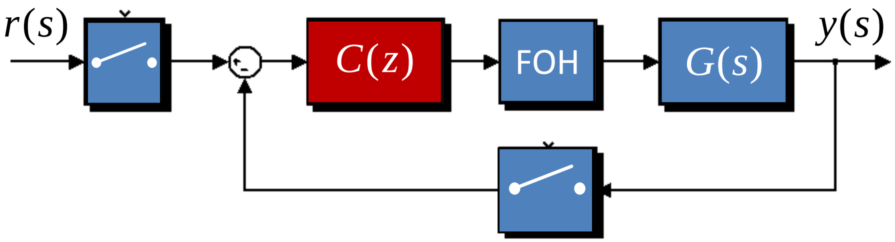
- Continuous TF doesn´t have a zero
- Discrete TF has a zero in

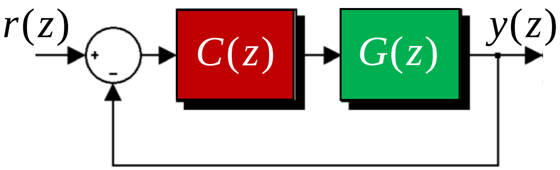
$$-\alpha = -\frac{e^{-amh} - e^{-ah}}{1-e^{-amh}}$$




```
>> m=0.1:0.01:1;alpha=(exp(-m)-exp(-1))./(1-exp(-m)); plot(m,-alpha);
>> syms m; m_sb=solve('(exp(-m)-exp(-1))/(1-exp(-m))=1')
m_sb = -log(1/2*exp(-1)+1/2)
>> vpa(m_sb,3)     ans = .380
```

Example





- For continues TF $G(s)=\dfrac{1}{s^2}$ we calculate

$$\mathcal{Z}\left\{\frac{G(s)}{s^2}\right\}=\mathcal{Z}\left\{\frac{1}{s^4}\right\}=\frac{t^3}{3!}\to\frac{(kh)^3}{6}\to\mathcal{Z}\left\{\frac{1}{6}(kh)^3\right\}=\frac{h^3}{6}\mathcal{Z}\left\{k^3\right\}=\frac{h^3}{6}\frac{z(z^2+4z+1)}{(z-1)^4}$$

- Then discrete TF is

$$G(z)=\frac{(z-1)^2}{hz}\frac{h^3}{6}\frac{z(z^2+4z+1)}{(z-1)^4}$$

$$=\frac{h^2}{6}\frac{(z^2+4z+1)}{(z-1)^2}$$

```
>> Gz=c2d(tf(1/s^2),1,'foh')
Transfer function:
0.1667 z^2 + 0.6667 z + 0.1667
------------------------------
        z^2 - 2 z + 1
Sampling time: 1
>> Gzp=sdf(Gz)
Gzp =
    0.1667(z+3.7321)(z+0.2679)
    --------------------------
          (z-1)(z-1)
```

Example

```
>> D=5/(s+5);DD=zpk(D); T=1/15;
>> DDtustin=c2d(DD,T,'tustin'),
   DDmpz=c2d(DD,T,'matched'), ...
   DDzoh=c2d(DD,T,'zoh'),DDfoh=c2d(DD,T,'foh'), ...
Zero/pole/gain:
          0.14286 (z+1)
          -------------             Sampling time: 0.066667
          (z-0.7143)
Zero/pole/gain:
          0.28347
          ----------                Sampling time: 0.066667
          (z-0.7165)
Zero/pole/gain:
          0.28347
          ----------                Sampling time: 0.066667
          (z-0.7165)
Zero/pole/gain:
          0.14959 (z+0.8949)
          ------------------        Sampling time: 0.066667
          (z-0.7165)

>> bode(DD,DDtustin,DDmpz,DDzoh,DDfoh)
>> omegas=2*pi/T            omegas = 94.2478
```

Frequency characteristics comparison

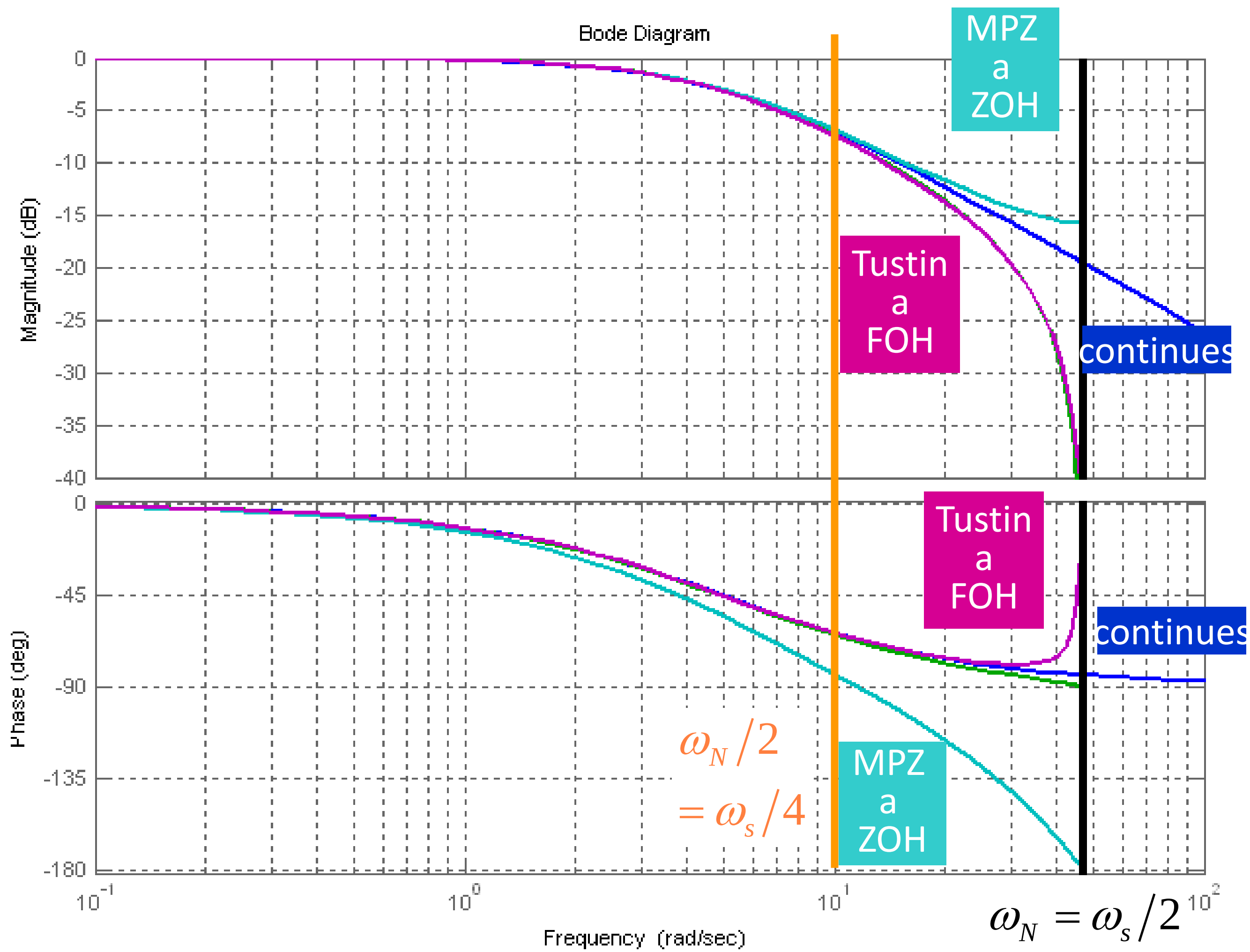
$$G(s)=\frac{5}{s+5}$$

$$h=1/15\ \text{s}$$

$$\omega_s\approx 94\,\text{rad}$$

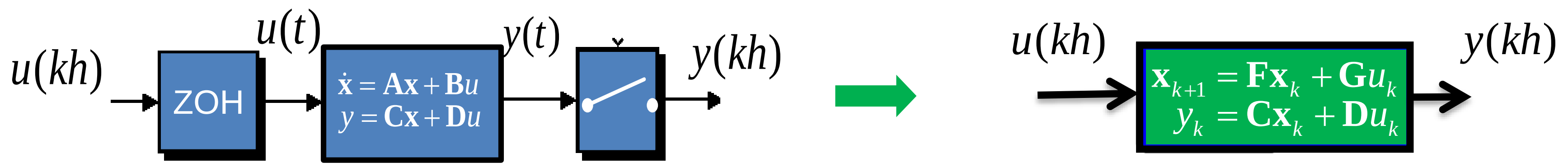
- All OK to

$$\omega_N/2=\omega_s/4$$



Discretization of the state model: Derivation

Discrete state model + 0. order shaping part



- We come out of the continuous model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u$$

- If the system in time t_0 has a state $x(t_0)$ then for $t \geq t_0$ the state is:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

- We need to know the input over the interval $[t_0, t)$

Discretization of the state model: Derivation

- We are interested in relation between state in time t_{k+1} and state in time t_k considering ZOH i.e. constant input
- $u_k = u(\tau), \tau \in [t_k, t_{k+1})$ During the sampling interval
- consider $h = t_{k+1} - t_k$ and we get

$$\mathbf{x}(t_{k+1}) = e^{\mathbf{A}(t_{k+1}-t_k)}\mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)}\mathbf{B}u(\tau)d\tau$$

$$= e^{\mathbf{A}h}\mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)}d\tau\mathbf{B}u(t_k) \qquad \nu = t_{k+1} - \tau$$

$$= e^{\mathbf{A}h}x(t_k) + \left(\int_0^h e^{\mathbf{A}\nu}d\nu\right)\mathbf{B}u(t_k)$$

- Sampling does not change the output equation so we get

$$\mathbf{x}(t_{k+1}) = \mathbf{F}\mathbf{x}(t_k) + \mathbf{G}u(t_k)$$
$$y(t_k) = \mathbf{C}\mathbf{x}(t_k) + Du(t_k)$$

$$\mathbf{F} = e^{\mathbf{A}h}$$
$$\mathbf{G} = \left(\int_0^h e^{\mathbf{A}\nu}d\nu\right)\mathbf{B}$$

Matrix exponential calculation

There are many methods for matrix exp. calculation

$$\mathbf{F} = e^{\mathbf{A}h}, \quad \mathbf{G} = \left(\int_0^h e^{\mathbf{A}\nu} d\nu\right)\mathbf{B}$$

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

- Decomposition in the Taylor series

$$\mathbf{V} = \int_0^h e^{\mathbf{A}\nu} d\nu = \mathbf{I}h + \frac{\mathbf{A}h^2}{2!} + \frac{\mathbf{A}^2 h^3}{3!} + \cdots + \frac{\mathbf{A}^i h^{i+1}}{(i+1)!} + \cdots \qquad \mathbf{F} = \mathbf{I} + \mathbf{A}\mathbf{V}, \quad \mathbf{G} = \mathbf{V}\mathbf{B}$$

- Jordan form (eigenvalues)

$$A = V \operatorname{diag}\{\lambda_i\} V^{-1} \;\Rightarrow\; e^{Ah} = V \operatorname{diag}\{e^{h\lambda_i}\} V^{-1}$$

>> **expmdemo2**

- Caylay-Hamilton theorem

>> **expmdemo3**

- Matlab function **`expm`** – Pade aproximation

>> **expmdemo1**

Example

- Continues 1st order system

$$\dot{x} = \alpha x + \beta u$$

- With sampling period h is

$$\mathbf{F} = e^{\mathbf{A}h} = e^{\alpha h}$$

$$\mathbf{G} = \left(\int_0^h e^{\alpha v} dv \right) \beta = \frac{\beta}{\alpha} \left(e^{\alpha v} - 1 \right)$$

- So the discrete system with ZOH is

$$x(k+1) = e^{\alpha h} x(k) + \frac{\beta}{\alpha} \left(e^{\alpha v} - 1 \right) u(k)$$

Example

- For double integrator

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- With sampling period h is

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

$$\mathbf{F} = e^{\mathbf{A}h} = \mathbf{I} + \mathbf{A}h + \mathbf{A}^2 h^2/2 + \ldots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} + 0 + \cdots = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{G} = \int_0^h e^{A\nu} \mathbf{B} d\nu = \int_0^h \begin{bmatrix} \nu \\ 1 \end{bmatrix} d\nu = \begin{bmatrix} h^2/2 \\ h \end{bmatrix}$$

- So the discrete system with ZOH is

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} h^2/2 \\ h \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

Immediate calculation problem

- In previous examples: numerator degree in z = denumerator degree in z
- Therefore differential controller equation
  $u(k)$ + parts with k-1,… = $ce(k)$ + parts with k-1, k-2,…
- Such a digital controller must compute immediately ie the delay resulting from the non-zero calculation time is neglected.
- This is practically acceptable only if the calculation
- time is < 1. Controller has at least one step delay
- numerator degree in z < denumerator degree in z
- We can get it by MPZ incompleteness
- Or we have to „add“ delay to system
- This is not a time delay (only way of indexing)