23 – Discrete systems

Michael Šebek
Automatic control 2016
Systems using radar
- Measuring the target position once per revolution
- Motivation at the beginning of discrete models

Analytical measuring instruments
- Variable parameters measure off-line by analytical instruments (mass spectrometer, chromatograph)

Economic systems
- Accounting procedures in economic systems are often tied to the calendar (date)
- Processes can run at any time, but they are charged (data is statistically computed) per day, week, month, ...
- Balance on account, profit, cost, exchange rate, share price, production, warehouse status, ...
Pulse-operated systems or actuators

Thyristor control
• Power electronics with thyristors work in pulses

Biological systems
• Signals in the nervous system are pulses

Combustion engines
• Ignition generates torque pulse (clock synchronizing motor)
• Classic rotary internal combustion engine (requires crankshaft)

• New principle: linear combustion engine, Department of Control Engineering: doc. Vysoký
Particle accelerator

- Dutch engineer Simon van der Meer has greatly improved the accelerator by introducing FB into position tracking
- This allowed to increase the intensity and improve the quality of the beam, which was a key factor in a successful CERN experiment that led to the discovery of W and B boson particles, mediating weak force
- The method was called stochastic cooling
- Van der Meer and Carlo Rubia received the Nobel Prize in Physics 1984
- Particles can only be seen in the sensor = sampling in the sensor
- "Push" is only available in the "kicker" = sampling in the actuator
Applications in computer disciplines

- IBM Lotus Domino Control, Queuing Systems, Roaming Overload Detection, Streaming Media, and more, see the IBM Book of Authors
- Web server management (Apache web server)
- Finite automata monitors processes and responds to pending requests
- For fast response to Web requests, neither the computational capacity nor the depleted memory should be overloaded - the feedback control algorithm
- 2 outputs and 2 references: processor load, memory usage
- 2 action steps = parameters change
  \[ \text{MaxClients} = \text{Maximum number of simultaneous-served requests} \]
  \[ \text{KeepAlive} \quad \text{The maximum time that an connection is maintained until interrupted} \]
- Working point \( x_{cpu} = 0.58, x_{mem} = 0.55, u_{mc} = 600, u_{ka} = 11 \text{s} \), with linearization
- State space model and matrix of transfer functions

\[
\begin{bmatrix}
x_{cpu}(k+1) \\
x_{mem}(k+1)
\end{bmatrix} =
\begin{bmatrix}
0.54 & -0.11 \\
-0.026 & 0.63
\end{bmatrix}
\begin{bmatrix}
x_{cpu}(k) \\
x_{mem}(k)
\end{bmatrix} +
\begin{bmatrix}
-0.0085 & 0.00044 \\
-0.00025 & 0.00028
\end{bmatrix}
\begin{bmatrix}
u_{ka}(k) \\
u_{mc}(k)
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_{cpu}(z) \\
x_{mem}(z)
\end{bmatrix} =
\begin{bmatrix}
0.0054 - 0.0085z \\
0.34 - 1.2z + z^2
\end{bmatrix}
\begin{bmatrix}
0.00036 - 0.0002z \\
0.34 - 1.2z + z^2
\end{bmatrix}^{-1}
\begin{bmatrix}
0.34 - 1.2z + z^2 \\
0.34 - 1.2z + z^2
\end{bmatrix}
\begin{bmatrix}
u_{ka}(z) \\
u_{mc}(z)
\end{bmatrix}
\]
Discrete state model and its solution

- **Discrete state-space (time invariant) model**

\[
x_{k+1} = Fx_k + Gu_k, \quad x_0
\]

\[
y_k = Hx_k + Ju_k
\]

- **Solution**

\[
x_1 = Fx_0 + Gu_0
\]

\[
x_2 = Fx_1 + Gu_1 = F(Fx_0 + Gu_0) + Gu_1 = F^2x_0 + FGu_0 + Gu_1
\]

\[\vdots\]

\[
x_k = F^kx_0 + \sum_{j=0}^{k-1} F^{k-j-1}Gu_j
\]

- **State-space matrix of transitions:**

\[
\Phi(k) = F^k
\]
State-space and external description

- State-space description if discrete model
  \[ x_{k+1} = Fx_k + Gu_k, \quad x_0 \]
  \[ y_k = Hx_k + Ju_k \]

- External description in \( z \)
  \[ H(zI - F)^{-1}G + J = \frac{b(z)}{a(z)} \]
  \[ zH(zI - F)^{-1}x_0 = \frac{c_{x_0}(z)}{a(z)} \]

- External description in \( z^{-1} = d \)
  \[ H(I - dF)^{-1}Gd + J = \frac{\hat{b}(d)}{\hat{a}(d)} \]
  \[ H(I - dF)^{-1}x_0 = \frac{\hat{c}_{x_0}(d)}{\hat{a}(d)} \]

- State realization are found in the same way as in a continuous case

- \( z \)-Transformation
  \[ \{x\} = x(z) = \sum_{k=0}^{\infty} x_k z^{-k} \]
  \[ \{x_{k+1}\} = zx(z) - zx_0 \]
Transfer function in $z$:
- The "physical" transf. fun. in $z$ is strictly proper
- for $n = m$ reacts immediately (counts $\infty$ fast)
- for $n < m$ predict the future (not causal)

To the transfer function in $d = z^{-1}$ it is projected differently $\hat{b}(d)/\hat{a}(d)$
- Properness correspond to causal denominator $\hat{a}(0) \neq 0$
- For strictly proper also $\hat{b}(0) = 0$

The order of the transfer function is as follows:
- TF in $z$: system order = order of the numerator (as in continuous case)
- TF in $d$: system order = $\max\left(\deg_d \hat{a}(d), \deg_d \hat{b}(z)\right)$

DC gain
\[
\frac{b(z)}{a(z)} = \frac{\hat{b}(z^{-1})}{\hat{a}(z^{-1})} \Rightarrow k_{DC} = \frac{b(1)}{a(1)} = \frac{\hat{b}(1)}{\hat{a}(1)}
\]
Poles and zeros

- Between the poles of continuous and sampled signal, e.g. the impulse response, applies:
  \[ z = e^{sh} = e^{(\alpha + j\omega)h} = e^{\alpha h} (\cos \omega h + j \sin \omega h) \]
- \( z \) has no unit, \( s \) (derivative operator) has unit \( 1/\text{čas} \)
- Stability boundary: The imaginary axis corresponds to the unit circle.
  \[ z = e^{j\omega h} = e^{j2\pi \omega / \omega_s} = e^{j\pi \omega / \omega_N} \]
- One whole circle matches the interval \( \omega \in [0, \omega_s], \omega_s = 2\pi / h = 2\omega_N \)
  Higher frequencies are overlapped with corresponding lower (aliasing)
- The negative real axis represents Nyquist frequencies
  \[ \alpha + j\omega_N, \omega_N = \omega_s / 2 = \pi / h \]
  specifically \( \alpha < 0 \rightarrow (-1,0), \alpha > 0 \rightarrow (-\infty,-1), \)
- The real axis corresponds to non-neg. real axis: \( \mathbb{R}^+ \rightarrow [1,\infty), \mathbb{R}^- \rightarrow (0,1) \)
- Dominant positions: point \( s = 0 \) corresponds to point \( z = 1 \)
- Non-significant positions: Real "far left" positions correspond to the "very close 0 right"
Effect of poles position

>> f = z/(1+z)
f = z / 1 + z
>> ft = f{0:-1:-10}
ft = 1 -1 1 -1 1 -1 1 -1 1
>> plot(0:1:length(ft)-1,ft)
>> picture(f,10)
Discrete Bode plot

- Complex exponential is a periodic function with a period of $2\pi$ and symmetric inside a period.
  \[ e^{j\omega h} = \cos \omega h + j \sin \omega h \]
- Amplitude of frequency transfer $G(z) = G(e^{j\omega h})$ is periodic function $\omega$ with period $0 \leq \omega \leq \omega_N = \omega_s/2 = \pi/h$ and symmetric inside period (on the linear scale $\omega$) Phase is "periodically offseted and antisymmetric"
- We draw the graph only for
  \[ \omega_s = 2\omega_N = 2\pi/h \]
  only for the upper half of the circle
- Can not draw it using asymptotes
- Sampling + shaping causes an additional phase delay ($e^{-\omega h/2}$)

\[ \Delta \phi = \angle G(j\omega) - \angle G_z(j\omega) = \frac{\omega h}{2} [\text{rad}] = \frac{180}{2\pi} \omega h [\text{deg}] \approx 29 \omega h [\text{deg}] \]
• \( G(z) = G(e^{j\omega h}) \) is a periodic function \( \omega \) with a period \( \omega_s = 2\omega_N = 2\pi/h \)
• Therefore Discrete Nyquist plot \( G(e^{j\omega h}) \) we usually draw for
  \[ 0 \leq \omega \leq \omega_N = \omega_s/2 = \pi/h \] (on the top half of the circle)
• Control System Tbx (default) draws it on whole circle \( -\omega_N \leq \omega \leq \omega_N \)

Example

\[
G(s) = \frac{1}{1+s}
\]

\[
G=1/(1+s);
nyquist(tf(G),c2d(tf(G),0.2),
c2d(tf(G),1),c2d(tf(G),2))
\]

\[
\omega = \pi/h
\quad \approx 15.7
\]

\[
Gz=c2d(tf(G),0.2),
nyquist(Gz)
\]

Transfer function:

\[
0.1813
\]

Sampling time: 0.2
Unlike a continuous case

Instability is outside the unit circle, it is not easy to circumscribe the contour, so, we surround the stability area.

Consider L strictly proper \( \rightarrow H(z) = 1 + L(z) \) has eq. poles and zeros \( \Re \sigma = n \)

Denote

- \( Z \) ... the number of unstable CL poles
- \( P \) ... the number of unstable CL poles
- \( N \) ... the number of the circle of the critical point -1 in the same direction as the region (here usually against the clock)

The argument principle follows:

\[
N = (n - Z) - (n - P) = P - Z
\]

CL system has \( Z = P - N \) unstable poles

Nyquist criterion of stability:

CL system is stable \( \iff P = N \) (against the clock hands)

Special case:

If the OL system is stable, then the CL system is stable

\( \iff \) Nyquist plot \( L(s) \) does not circle critical point -1

Discrete Nyquist Criterion

Continuous - for comparison

\[
\begin{align*}
N &= Z - P \\
Z &= N + P \\
P &= -N
\end{align*}
\]

but also against the clock hands
Sung a Hara (1988)

For a system where $L(z)$ has $n_p$ unstable poles $p_i = r_i e^{j\phi_i}$, $r_i > 1$ it holds the limitation

$$\int_0^\pi \ln |S(e^{j\omega})| d\omega = \pi \sum_{0}^{n_p} \ln r_i$$

cont. case for comparison

$$\int_0^\infty \ln |S(j\omega)| d\omega = \pi \sum_{0}^{n_p} \text{Re} p_i$$

Differences to continuous version:
• there is no condition of relative order
• the integral is over the final interval, therefore we can only spill over this final frequency interval