24 – Discrete control

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Discrete Control Design Methods

Design for purely discrete systems

- Many methods are analogous (or identical) to continuous methods, so we do not mention them, just show them on examples.
- These methods are briefly described in additional presentation.
- Here we focus on differences in a discrete design.

Discrete Control Design for Sampled Continuous Systems

- We come out of a discrete system model and use discrete design methods.
- CL stability is guaranteed (by a stabilizing discrete controller), unlike the emulation methods.
- Control works well at sampling times (within a reasonable period of time)
- On the contrary, we do not control the behavior between samples.
- Behavior between sampling times is fine unless action is "very wild"

Deadbeat

- Typically discrete control design strategy, continuous (not exactly) work
- Goal = all poles „to zero“ i.e. $p_{\text{new}}(z) = z^n$
  so for the result holds (Cayley-Hamilton) $\mathbf{F}_{\text{new}}^n = 0$
- Because equation $\mathbf{x}_{k+1} = \mathbf{F}_{\text{new}} \mathbf{x}_k$ has solution $\mathbf{x}_k = \mathbf{F}_{\text{new}}^k \mathbf{x}_0$,
  and $\mathbf{x}_k = 0, \forall k \geq n, \forall \mathbf{x}_0$
- Thus, regardless of the initial state, the system is completely at the steady state with n-th step (all states and output are zero)!
- In case of disturbance, it also gets to steady state (on latest in the n-th step after the disturbance has vanished)
- The resulting system is valid for

$$\mathbf{x}(z) = \left(z\mathbf{I} - \mathbf{F}_{\text{new}}\right)^{-1} \mathbf{G} u(z) + z\left(z\mathbf{I} - \mathbf{F}_{\text{new}}\right)^{-1} \mathbf{x}_0 = \frac{\operatorname{adj}\left(z\mathbf{I} - \mathbf{F}_{\text{new}}\right)\mathbf{G}}{z^n} u(z) + \frac{z \operatorname{adj}\left(z\mathbf{I} - \mathbf{F}_{\text{new}}\right)}{z^n} \mathbf{x}_0$$

$$y(z) = \frac{\mathbf{H} \operatorname{adj}\left(z\mathbf{I} - \mathbf{F}_{\text{new}}\right)\mathbf{G}}{z^n} u(z) + \frac{z\mathbf{H} \operatorname{adj}\left(z\mathbf{I} - \mathbf{F}_{\text{new}}\right)}{z^n} \mathbf{x}_0 = \frac{b(z)}{z^n} u(z) + \frac{c_{x_0}(z)}{z^n}$$

$$\deg\left(b(z), c_x(z)\right) \leq n$$

Deadbeat

- The deadbeat system is the most stable of all discrete ones.
  For continuous systems holds the same: No later than n steps, from each initial state is at steady state – i.e. all states and output are zero. even between sampling times!   Continuous control doesn't allow that!
- The deadbeat system responds very quickly: sometimes it is beneficial but sometimes disadvantageous (if there is noise in the system).
- Shortening sampling period, size of input signals increases, and limit of h → 0 goes to infinity. Therefore, we do not reduce the period.
- Deadbeat is designed just like any other pole assignment.
- Because $p_{new}(z) = z^n$, is Ackermann's formula specially

$$\mathbf{K} = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \mathbf{F}^n$$

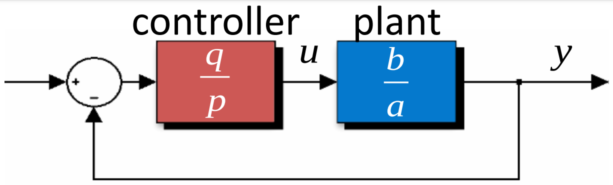
- If matrix $\boldsymbol{F}$ is invertible, it holds that

$$\mathbf{K} = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{F}^{-n}\mathbf{G} & \mathbf{F}^{-n+1}\mathbf{G} & \cdots & \mathbf{G} \end{bmatrix}^{-1}$$

Deadbeat observer

- Even when estimating states by a discrete observer, we can use the deadbeat strategy.
- We choose the characteristic polynomial of the observation dynamics $p_{\mathrm{poz}}(z) = z^n$, so $\mathbf{F}_{\mathrm{poz}}^n = 0$
- For the observation error $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$, where $\tilde{\mathbf{x}}_{k+1} = \mathbf{F}_{\mathrm{poz}} \tilde{\mathbf{x}}_k$ it holds that $$\tilde{\mathbf{x}}_k = 0, \forall k \geq n, \forall \tilde{\mathbf{x}}_0$$
- Thus latest at n-th step, the deviation is zero, and from that moment on, the observer status is exactly equal to the state of the system.
- If the system is continuous, the same applies, but only at samples.
- Such an observer will be proposed, for example, by modifying the Ackermann's formula $$\mathbf{L} = \mathbf{F}^n \mathcal{O}^{-1} \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T$$

Polynomial Pole Assignment



Polynomial solution in *z* – same as continuous

- For given a system $b(z)/a(z)$

  and given pole positions, given by CL characteristic pol. $c(z)$
- we solve the equation $a(z)p(z)+b(z)q(z)=c(z)$

Polynomial solution in *d*

- Similarly $b(d)/a(d),c(d)\Rightarrow a(d)p(d)+b(d)q(d)=c(d)\Rightarrow q(d)/p(d)$

Deadbeat polynomial – special case of pole assignment

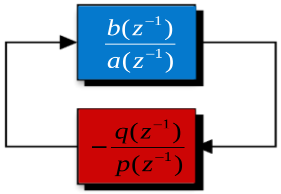
- In *z* we choose $c(z)=z^m$, where m ≥ (2× system order) – 1 , solve

$$a(z)p(z)+b(z)q(z)=z^m$$

  and choose the solution of minimum degree q
- By solving in z^{-1} is the solution simpler: We solve the equation

$$a(z^{-1})p(z^{-1})+b(z^{-1})q(z^{-1})=1$$

Stabilization by discrete controller



- If we work in z, then the parameterization of all the stabilizing regulators is the same as in cont. time.
- If we work in z^{-1}, it is even simpler:

All stabilizing controllers are parametrized as follows

$$\frac{q}{p} = \frac{y + at}{x - bt}$$

- Where t is any fraction of polynomials with a stable denominator and polynomials x, y fulfill $\bar{a}x + \bar{b}y = 1$
- where $a = (a,b)\bar{a},\ b = (a,b)\bar{b}$

Solvability:

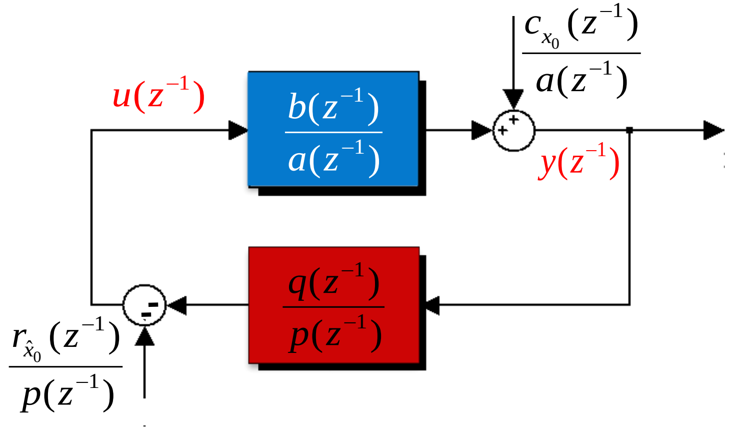
- The system does not have unstable hidden modes and $\gcd(a,b)$ is stable.

Deriving a weak and a strong version of deadbeat

- The design of deadbeat regulator in z^{-1} is very similar to that in z
- So, for the sake of interest, at least choose the opposite derivation:
- We have considered deadbeat to be a special case of poles (When all poles are placed to origin, the behavior will be deadbeat)
- Now, look for a regulator so that the behavior is of the deadbeat type:
- In addition, we will work with polynomials in z^{-1} (it is more convenient).

Problem formulation: Deadbeat controller – strong version

- Find the controller so that the input and output sequences have a final length for each i.c. of plant and controll
- Moreover, in order to have the shortest length (the smallest number of steps).



Derivation: Deadbeat – strong version

- We derive from the equations of the system

$$y = -\frac{b}{ap+bq} r_{\hat{x}_0} + \frac{p}{ap+bq} c_{x_0}$$

$$u = -\frac{a}{ap+bq} r_{\hat{x}_0} - \frac{q}{ap+bq} c_{x_0}$$

- The goal is to find polynomials $p(z^{-1}), q(z^{-1})$ such, that for all possible polynomials $r_{\hat{x}_0}(z^{-1}), c_{x_0}(z^{-1})$ the resulting sequences were polynomials (= had a final length).
- Because $r_{\hat{x}_0}, c_{x_0}$ are not give (they represent different init. cond.), each of the four fractions must be polynomial in itself.
- A polynomial fraction is generally an infinite sequence. The endpoint will be when the denominator divides the numerator without the rest.
- However, we can show that we can not simultaneously denote the denominator $ap+bq$ with all four numerators a, b, p, q
- Each of the fractions is a polynomial $\Leftrightarrow$ $a(z^{-1})p(z^{-1}) + b(z^{-1})q(z^{-1}) = 1$

Derivation: Deadbeat – strong version

- It holds that $ap + bq = 1$, then the above relationships are given

$$y(z^{-1}) = b(z^{-1})r_{\hat{x}_0}(z^{-1}) + p(z^{-1})c_{x_0}(z^{-1})$$
$$u(z^{-1}) = a(z^{-1})r_{\hat{x}_0}(z^{-1}) - q(z^{-1})c_{x_0}(z^{-1})$$

- Because everything on the right are polynomials, they are $y(z^{-1}), u(z^{-1})$ polynomial for all initial condition as well, which had to be ensured.
- The length of sequences is given by the degree of polynomials +1
- The only option for shortening in some i.c., $\left[p(z^{-1}), q(z^{-1}) \right]$ is to choose a solution of the minimum degree.

Summary of the solution

- To find a strong version of the deadbeat controller, you need to solve the polynomial equation $a(z^{-1})p(z^{-1}) + b(z^{-1})q(z^{-1}) = 1$ and from possible solutions to select the one with minimum degree $\left[p(z^{-1}), q(z^{-1}) \right]$
- Solvability: $\boldsymbol{a}, \boldsymbol{b}$ relatively prime (controllable and observable system), there must also be no hidden modes.

Deadbeat – weak version

Problem formulation: Deadbeat controller – weak version

- Find the controller so that the output sequence *y* has a final length, for each initial conditions for both, the system and the controller.
- Moreover, so that it has the shortest length.
- At the same time the CL system must be stable.
- Difference against strong version: *u* can be infinitely long but stable.

Solution

- We are only interested in one formula and then we can cancel against b.

$$y = -\frac{b}{ap+bq} r_{\hat{x}_0} + \frac{p}{ap+bq} c_{x_0}$$

- Divide b to stable and unstable factors $b(z^{-1}) = b^{+}(z^{-1}) b^{-}(z^{-1})$
- and set $ap + bq = b^{+}$

Deadbeat – weak version

- Equation $ap + bq = b^+$ can be divided by b^+ (or p must be divisible) we obtain $ax + b^- q = 1$, where $p = xb^+$
- Then the output relationship switches to $y = b^- r_{\hat{x}_0} + xc_{x_0}$
- Thus, the output is a polynomial for all initial conditions.
- The shortest march we get by choosing a solution with x min. degree.
- We'll check the input u: (it is not finite but it is stable)

$$u = -\frac{a}{ap+bq} r_{\hat{x}_0} - \frac{q}{ap+bq} c_{x_0} = -\frac{a}{b^+} r_{\hat{x}_0} - \frac{q}{b^+} c_{x_0}$$

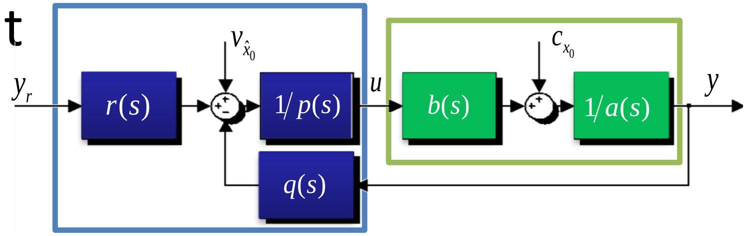
- CL system is stable as well, or $ap + bq = b^+$

Solution summary

- Divide $b(z^{-1}) = b^+(z^{-1}) b^-(z^{-1})$
- Solve equation $a(z^{-1})x(z^{-1}) + b^-(z^{-1})q(z^{-1}) = 1$ for x min. degree.
- Solvability: a, b^- relatively prime $\gcd(a, b^-)$ stable.

Controller 2DOF

- If the control system has a reference input it is natural to use a controller with two degrees of freedom (2DOF)



$$u(z) = -\frac{q(z)}{p(z)}y(z) + \frac{r(z)}{p(z)}y_r(z) \qquad p(z)u(z) = -q(z)y(z) + r(z)y_r(z)$$

- For causality must be $\deg p \geq \deg\left[q(z), r(z)\right]$
- Classic deviation control (1DOF) is a special case when $q(z) = r(z)$
- We compute the FB part of the known equation when designing

$$a(z)p(z) + b(z)y(z) = c(z)$$

- Where we suitably choose a CL characteristic polynomial
- From a comparison with a state approach it follows that $c(z) = c_c(z)c_o(z)$
- Where the factors are $c_c(z) = \det\left(zI - F + GK\right), c_o(z) = \det\left(zI - F + LH\right)$

Direct branch

- The resulting system transfer function is

$$y(z)=\frac{b(z)r(z)}{a(p)p(z)+b(z)q(z)}y_r(z)$$
$$=\frac{b(z)r(z)}{c(z)}y_r(z)=\frac{b(z)r(z)}{c_c(z)c_o(z)}y_r(z)$$

- We choose the direct branch to cancel the poles of the observer, i.e. $c_o(z)\,|\,r(z)$ for example as

$$r(z)=t_0c_o(z)$$

$$y(z)=\frac{t_0b(z)}{c_c(z)}y_r(z)$$

- Then the control do not to generate a deviation of the observations
- constant t_0 is chosen to ensure required static gain
- Usually, the static gain should be = 1, therefore we set $t_0=c_c(1)/b(1)$

Discreet tracking - asymptotic and deadbeat

Asymptotic tracking is the same as for continuous systems

- Equations are the same

$$ap + bq = m, \quad f^{-}t + br = m, \quad m \text{ and stable}$$

- Conditions are the same

  1) $\gcd(a,b)$ stable; 2) $\gcd(f^{-},b)=1$; 3) $f^{-} \mid a$

- Solution is the same in z and in z^{-1}, Except that we still have to choose *m* of appropriately high degree when dealing with z

  Unlike in a continuous case, there is a deadbeat option, which allows tracking in the final number of steps:

- If we proceed in z, we choose $m(z) = z^{n-1}$

  if in z^{-1}, we choose $m(z^{-1}) = 1$

- And choose the solution of minimum degrees (coincidence occurs)
- Solution exists, iff $\gcd(a,b) = 1$

  the other conditions are the same.