Design for purely discrete systems

- Many methods are analogous (or identical) to continuous methods, so we do not mention them, just show them on examples.
- These methods are briefly described in additional presentation.
- Here we focus on differences in a discrete design.

Discrete Control Design for Sampled Continuous Systems

- We come out of a discrete system model and use discrete design methods.
- CL stability is guaranteed (by a stabilizing discrete controller), unlike the emulation methods.
- Control works well at sampling times (within a reasonable period of time)
- On the contrary, we do not control the behavior between samples.
- Behavior between sampling times is fine unless action is "very wild"
Typically discrete control design strategy, continuous (not exactly) work

Goal = all poles „to zero“ i.e. 

\[ p_{\text{new}}(z) = z^n \]

so for the result holds (Cayley-Hamilton) 

\[ F_n^{\text{new}} = 0 \]

Because equation 

\[ x_{k+1} = F_{\text{new}} x_k \]

has solution 

\[ x_k = F_{\text{new}}^k x_0 \]

and \( x_k = 0, \forall k \geq n, \forall x_0 \)

Thus, regardless of the initial state, the system is completely at the steady state with \( n \)-th step (all states and output are zero)!

In case of disturbance, it also gets to steady state (on latest in the \( n \)-th step after the disturbance has vanished)

The resulting system is valid for

\[
x(z) = (zI - F_{\text{new}})^{-1} Gu(z) + z(zI - F_{\text{new}})^{-1} x_0 = \frac{\text{adj}(zI - F_{\text{new}})G}{z^n} u(z) + \frac{z \text{adj}(zI - F_{\text{new}})}{z^n} x_0
\]

\[
y(z) = \frac{H \text{adj}(zI - F_{\text{new}})G}{z^n} u(z) + \frac{zH \text{adj}(zI - F_{\text{new}})}{z^n} x_0 = \frac{b(z)}{z^n} u(z) + \frac{c_x(z)}{z^n}
\]

\[ \deg(b(z), c_x(z)) \leq n \]
• The deadbeat system is the most stable of all discrete ones. For continuous systems holds the same: No later than n steps, from each initial state is at steady state – i.e. all states and output are zero, even between sampling times! Continuous control doesn’t allow that!

• The deadbeat system responds very quickly: sometimes it is beneficial but sometimes disadvantageous (if there is noise in the system).

• Shortening sampling period, size of input signals increases, and limit of $h \to 0$ goes to infinity. Therefore, we do not reduce the period.

• Deadbeat is designed just like any other pole assignment.

• Because $p_{\text{new}}(z) = z^n$, is Ackermann’s formula specially

$$K = [0 \ldots 0 1] C^{-1} F^n$$

• If matrix $F$ is invertible, it holds that

$$K = [0 \ldots 0 1] [F^{-n} G \ F^{-n+1} G \ \ldots \ G]^{-1}$$
• Even when estimating states by a discrete observer, we can use the deadbeat strategy.

• We choose the characteristic polynomial of the observation dynamics 
  \[ p_{\text{poz}}(z) = z^n \], so 
  \[ F^n = 0 \]

• For the observation error \( \hat{X} = x - \hat{x} \), where 
  \[ \tilde{x}_{k+1} = F_{\text{poz}} \tilde{x}_k \]
  it holds that 
  \[ \tilde{x}_k = 0, \ \forall k \geq n, \ \forall \tilde{x}_0 \]

• Thus latest at n-th step, the deviation is zero, and from that moment on, the observer status is exactly equal to the state of the system.

• If the system is continuous, the same applies, but only at samples.

• Such an observer will be proposed, for example, by modifying the Ackermann’s formula 
  \[ L = F^n O^{-1} [0 \ldots 0 1]^T \]
Polynomial solution in $z$ – same as continuous

• For given a system $b(z)/a(z)$ and given pole positions, given by CL characteristic pol. $c(z)$

• we solve the equation $a(z)p(z) + b(z)q(z) = c(z)$

Polynomial solution in $d$

• Similarly $b(d)/a(d), c(d) \Rightarrow a(d)p(d) + b(d)q(d) = c(d) \Rightarrow q(d)/p(d)$

Deadbeat polynomial – special case of pole assignment

• In $z$ we choose $c(z) = z^m$, where $m \geq (2 \times \text{system order}) - 1$, solve

$$a(z)p(z) + b(z)q(z) = z^m$$

and choose the solution of minimum degree $q$

• By solving in $z^{-1}$ is the solution simpler: We solve the equation

$$a(z^{-1})p(z^{-1}) + b(z^{-1})q(z^{-1}) = 1$$
• If we work in $z$, then the parameterization of all the stabilizing regulators is the same as in cont. time.

• If we work in $z^{-1}$, it is even simpler:

All stabilizing controllers are parametrized as follows

\[
\frac{q}{p} = \frac{y + at}{x - bt}
\]

• Where $t$ is any fraction of polynomials with a stable denominator and polynomials $x, y$ fulfill $\bar{a}x + \bar{b}y = 1$

• where $a = (a, b)\bar{a}, \ b = (a, b)\bar{b}$

Solvability:

• The system does not have unstable hidden modes and $\gcd(a, b)$ is stable.
Deriving a weak and a strong version of deadbeat

- The design of deadbeat regulator in $z^{-1}$ is very similar to that in $z$
- So, for the sake of interest, at least choose the opposite derivation:
- We have considered deadbeat to be a special case of poles (When all poles are placed to origin, the behavior will be deadbeat)
- Now, look for a regulator so that the behavior is of the deadbeat type:
- In addition, we will work with polynomials in $z^{-1}$ (it is more convenient).

**Problem formulation: Deadbeat controller – strong version**

- Find the controller so that the input and output sequences have a final length for each i.c. of plant and controller
- Moreover, in order to have the shortest length (the smallest number of steps).
Derivation: Deadbeat – strong version

- We derive from the equations of the system

\[ y = -\frac{b}{ap + bq} r_{x_0} + \frac{p}{ap + bq} c_{x_0} \]

\[ u = -\frac{a}{ap + bq} r_{\hat{x}_0} - \frac{q}{ap + bq} c_{x_0} \]

- The goal is to find polynomials \( p(z^{-1}), q(z^{-1}) \) such, that for all possible polynomials \( r_{x_0}(z^{-1}), c_{x_0}(z^{-1}) \) the resulting sequences were polynomials (= had a final length).

- Because \( r_{\hat{x}_0}, c_{x_0} \) are not give (they represent different init. cond.), each of the four fractions must be polynomial in itself.

- A polynomial fraction is generally an infinite sequence. The endpoint will be when the denominator divides the numerator without the rest.

- However, we can show that we can not simultaneously denote the denominator \( ap + bq \) with all four numerators \( a, b, p, q \)

- Each of the fractions is a polynomial \( a(z^{-1})p(z^{-1}) + b(z^{-1})q(z^{-1}) = 1 \)
Derivation: Deadbeat – strong version

• It holds that \( ap + bq = 1 \), then the above relationships are given
  \[
y(z^{-1}) = b(z^{-1})r_{x_0}(z^{-1}) + p(z^{-1})c_{x_0}(z^{-1})
  \]
  \[
u(z^{-1}) = a(z^{-1})r_{x_0}(z^{-1}) - q(z^{-1})c_{x_0}(z^{-1})
  \]
• Because everything on the right are polynomials, they are \( y(z^{-1}), u(z^{-1}) \)
polynomial for all initial condition as well, which had to be ensured.
• The length of sequences is given by the degree of polynomials +1
• The only option for shortening in some \( i.c. \), \[
\begin{bmatrix}
p(z^{-1}), q(z^{-1})
\end{bmatrix}
\]
is to choose a solution of the minimum degree.

Summary of the solution

• To find a strong version of the deadbeat controller, you need to solve
  the polynomial equation \( a(z^{-1})p(z^{-1}) + b(z^{-1})q(z^{-1}) = 1 \) and from
  possible solutions to select the one with minimum degree \[
\begin{bmatrix}
p(z^{-1}), q(z^{-1})
\end{bmatrix}
\]
• Solvability: \( a, b \) relatively prime (controllable and observable system),
  there must also be no hidden modes.
Problem formulation: Deadbeat controller – weak version

• Find the controller so that the output sequence $y$ has a final length, for each initial conditions for both, the system and the controller.
• Moreover, so that it has the shortest length.
• At the same time the CL system must be stable.
• Difference against strong version: $u$ can be infinitely long but stable.

Solution

• We are only interested in one formula and then we can cancel against $b$.
• Divide $b$ to stable and unstable factors $b(z^{-1}) = b^+(z^{-1})b^-(z^{-1})$
• and set $ap + bq = b^+$

$$y = -\frac{b}{ap + bq} r_{x_0} + \frac{p}{ap + bq} c_{x_0}$$
• Equation $ap + bq = b^+$ can be divided by $b^+$ (or $p$ must be divisible) we obtain $ax + b^- q = 1$, where $p = xb^+$

• Then the output relationship switches to $y = b^- r_{\hat{x}_0} + xc_{x_0}$

• Thus, the output is a polynomial for all initial conditions.

• The shortest march we get by choosing a solution with $x$ min. degree.

• We'll check the input $u$: (it is not finite but it is stable)

\[
u = -\frac{a}{ap+bq} r_{\hat{x}_0} - \frac{q}{ap+bq} c_{x_0} = -\frac{a}{b^+} r_{\hat{x}_0} - \frac{q}{b^+} c_{x_0}\]

• CL system is stable as well, or $ap + bq = b^+$

**Solution summary**

• Divide $b(z^{-1}) = b^+(z^{-1})b^-(z^{-1})$

• Solve equation $a(z^{-1})x(z^{-1}) + b^-(z^{-1})q(z^{-1}) = 1$ for $x$ min. degree.

• Solvability: $a, b^-$ relatively prime $\gcd(a, b^-)$ stable.
If the control system has a reference input it is natural to use a controller with two degrees of freedom (2DOF)

\[ u(z) = -\frac{q(z)}{p(z)} y(z) + \frac{r(z)}{p(z)} y_r(z) \]

\[ p(z)u(z) = -q(z)y(z) + r(z)y_r(z) \]

For causality must be \( \text{deg } p \geq \text{deg } [q(z), r(z)] \)

Classic deviation control (1DOF) is a special case when \( q(z) = r(z) \)

We compute the FB part of the known equation when designing

\[ a(z)p(z) + b(z)y(z) = c(z) \]

Where we suitably choose a CL characteristic polynomial

From a comparison with a state approach it follows that \( c(z) = c_c(z)c_o(z) \)

Where the factors are \( c_c(z) = \det(zI - F + GK) \), \( c_o(z) = \det(zI - F + LH) \)
The resulting system transfer function is

\[
y(z) = \frac{b(z)r(z)}{a(p)p(z) + b(z)q(z)} y_r(z)
\]

\[
= \frac{b(z)r(z)}{c(z)} y_r(z) = \frac{b(z)r(z)}{c_c(z)c_o(z)} y_r(z)
\]

We choose the **direct branch** to cancel the poles of the observer, i.e. \(c_o(z) | r(z)\) for example as

\[
r(z) = t_0 c_o(z)
\]

Then the control do not to generate a deviation of the observations

constant \(t_0\) is chosen to ensure required static gain

Usually, the static gain should be \(= 1\), therefore we set \(t_0 = c_c(1)/b(1)\)
Asymptotic tracking is the same as for continuous systems

- Equations are the same
  \[ ap + bq = m, \quad f^- t + br = m, \quad m \text{ and } \text{stable} \]
- Conditions are the same
  1) \( \gcd(a, b) \) stable; 2) \( \gcd(f^-, b) = 1 \); 3) \( f^- \mid a \)
- Solution is the same in \( z \) and in \( z^{-1} \), Except that we still have to choose \( m \) of appropriately high degree when dealing with \( z \).
  Unlike in a continuous case, there is a deadbeat option, which allows tracking in the final number of steps:
  - If we proceed in \( z \), we choose \( m(z) = z^{n-1} \)
    if in \( z^{-1} \), we choose \( m(z^{-1}) = 1 \)
  - And choose the solution of minimum degrees (coincidence occurs)
  - Solution exists, iff \( \gcd(a, b) = 1 \)

the other conditions are the same.