NONLINEAR SYSTEMS

AUTOMATIC CONTROL, 2015

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TYPES OF NONLINEAR SYSTEMS

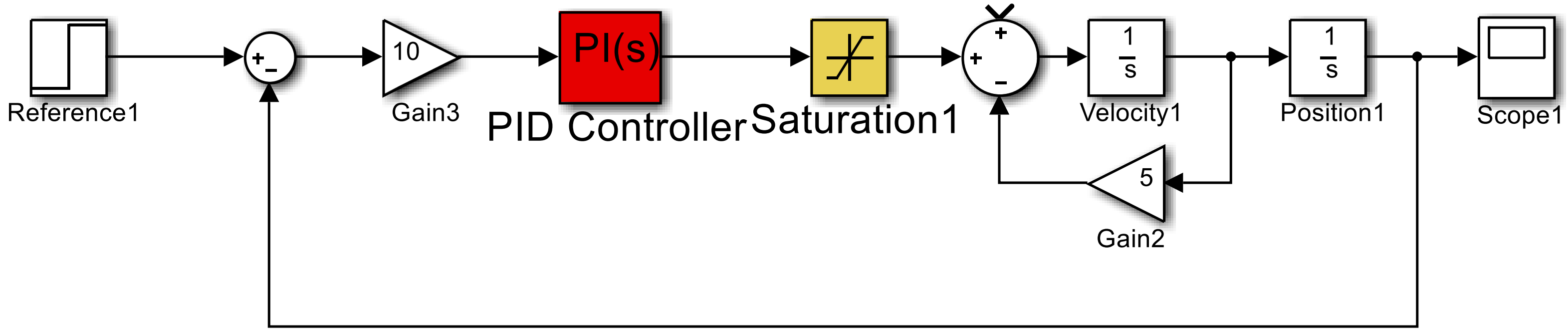
- Nonlinear systems
  - Mechanical systems with rotations - sines, cosines
  - Electric circuits – diodes, AC drives

  } Usually can be dealt with using linearization
- Nonlinear control
  - Bang-bang control
  - Relay

  } Simple control, complicated analysis
- Linear(ized) systems with static nonlinearity
  - Saturation, deadzone
  - Hysteresis, backlash
  - Nonlinear input characteristic

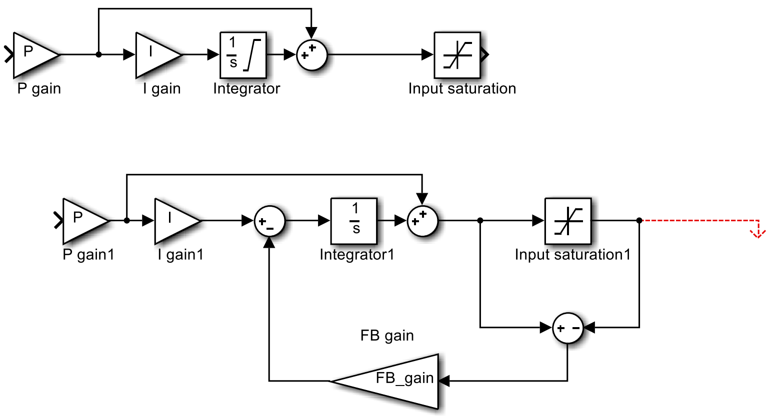
  } Cannot be linearized

SIMPLE EXAMPLE – LIMITED CONTROL EFFORT AND PI



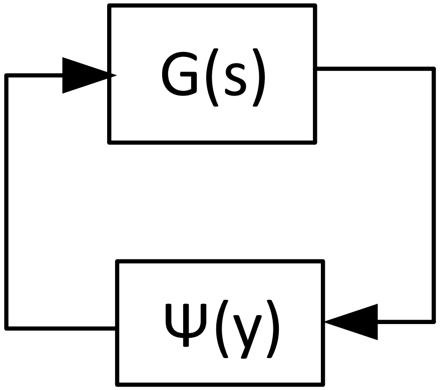
WIND-UP EFFECT

- Solution
  - Clamping – saturate the state of the
  - Feedback – back-calculation
- Always check your control effort!
- Design the P part to match the saturation.



STABILITY FOR STATIC NONLINEARITIES

- Linear dynamics, static nonlinear function
- Eigenvalue criterion does not work – often system is not linearizable
- Two typical types of behavior:
  - Stability/instability – Popov criterion, circle criterion
  - Limit cycles – harmonic analysis

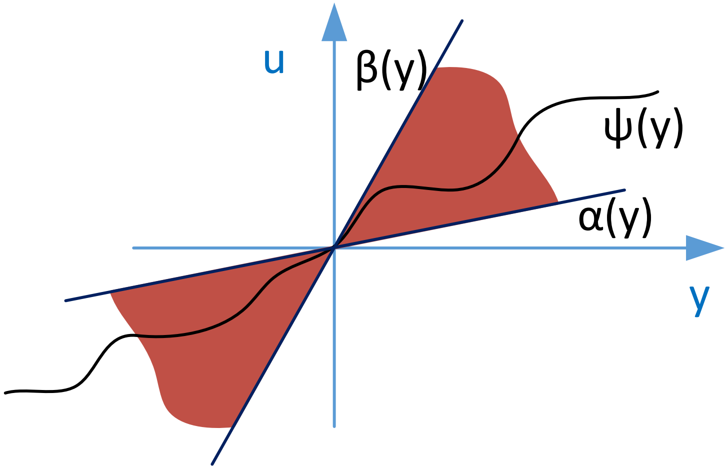


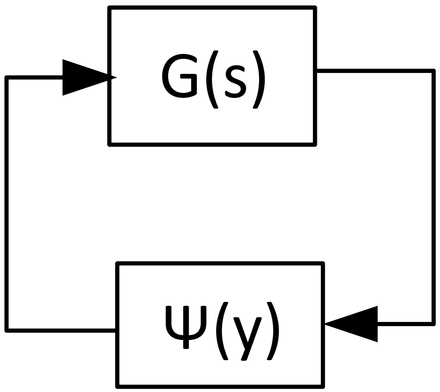
SYSTEMS WITH SECTOR NONLINEARITIES

- Memoryless function $u = \psi(y)$ is in the sector $[\alpha, \beta]$ if

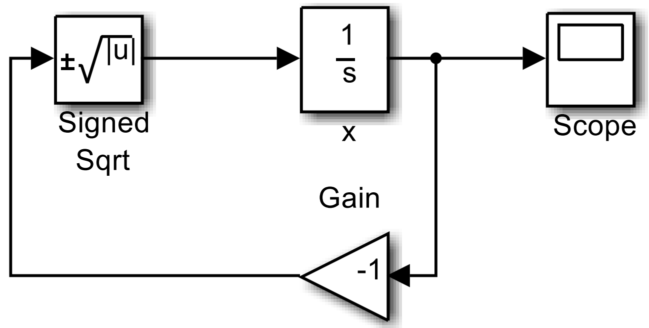
  $\alpha y \le \psi(y) \le \beta y$ for $y > 0$

  $\beta y \le \psi(y) \le \alpha y$ for $y \le 0$





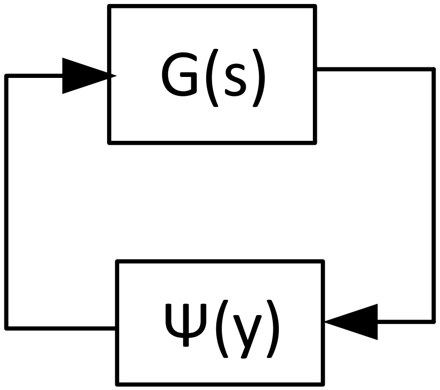
EXAMPLE



- Other examples:
  - Gain scheduling
  - Saturation + deadzone
- Does not include
  - Higher-order polynomials,
  - Exponentials, …

POPOV CRITERION

- Frequency based – modified frequency response
- Conditions for use:
  - Time invariant nonlinearity ψ in sector $[0, \beta]$
  - $\psi(0) = 0$
  - $G(s) = \frac{1}{s^n}\frac{p(s)}{q(s)}$ with $\deg(p(s)) < \deg(q(s))$
  - The poles $G(s)$ are in CLHP – left half plane or on imaginary axis
  - Marginally stable in singular case

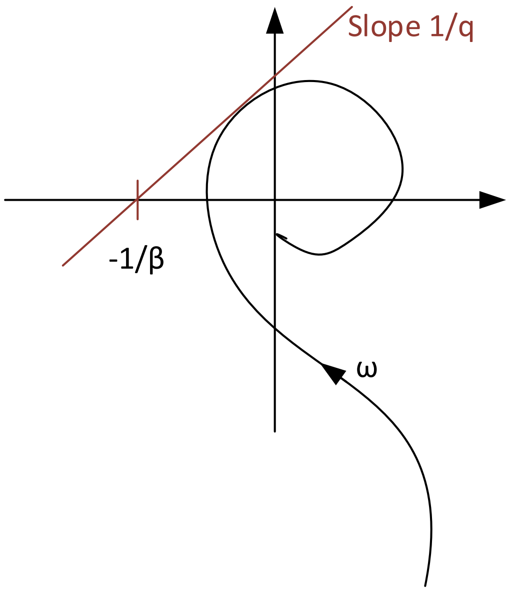


- Popov criterion: The closed loop is absolutely stable if for ψ in $[0, \beta]$ there exists a constant q such that

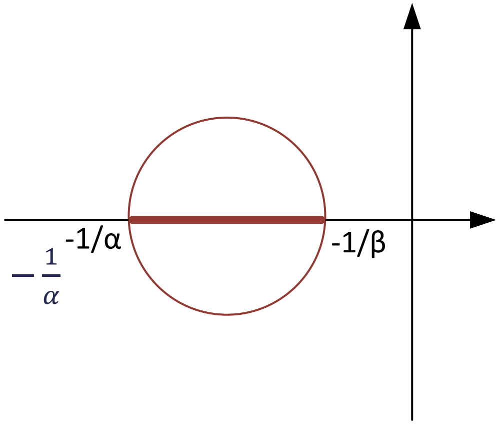
$$\Re\{G(j\omega)\} - qj\omega\Im\{G(j\omega)\} \succ \frac{1}{\beta}$$

POPOV CRITERION – NOTES

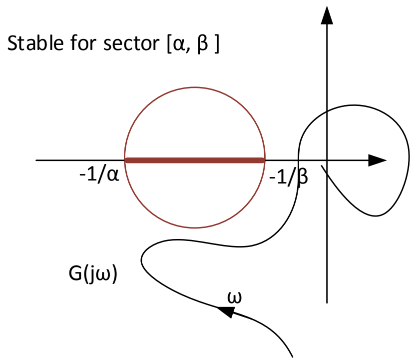
- Only for time-invariant nonlinearity
- Strong result – holds for arbitrary nonlinearity in the sector
- If sector condition holds on finite interval – finite domain of stability
- Only sufficient condition!

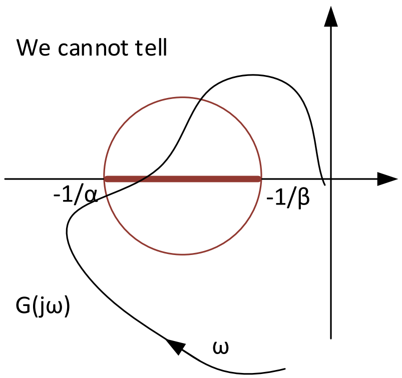


CIRCLE CRITERION



- Suitable for time-varying nonlinearity $\psi[\alpha, \beta]$
- Uses disk $D(\alpha, \beta)$ defined over the line segment on the real line between $-1/\beta$ and $-\frac{1}{\alpha}$
- Only sufficient condition
- The system is absolutely stable if either of the following holds:
  1. $0 < \alpha < \beta$: The Nyquist plot of G(s) does not enter the disk and encircles it m-times. m is the number of ORHP poles
  2. $0 = \alpha < \beta$: For stable G(s) the Nyquist plot stays to the right of $-1/\beta$
  3. $\alpha < 0 < \beta$: For stable G(s) the Nyquist plot stays within $D(\alpha, \beta)$

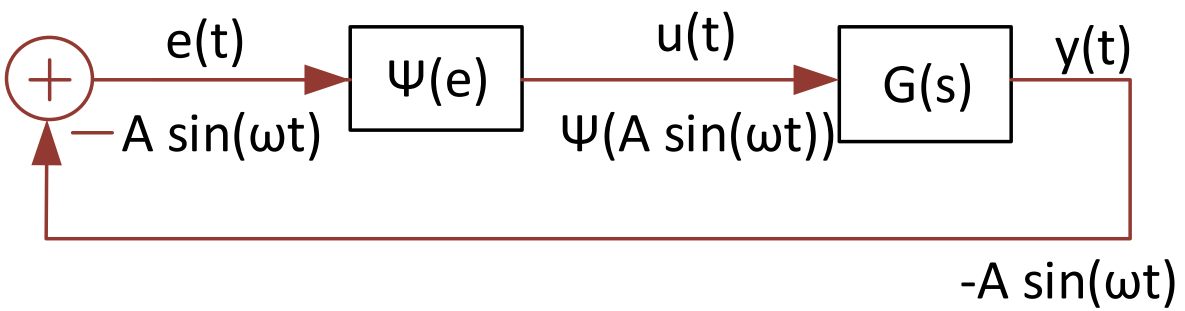




DESCRIBING FUNCTION METHOD

- Also known as harmonic balance
- Tests a presence of a limit cycle
- Assumes harmonic signal in the system
- Assumptions
  - Single nonlinearity, time invariant
  - System has low-pass properties
  - Nonlinearity symmetric to the origin

DESCRIBING FUNCTIONS



- If harmonic oscillations $e(t) = Asin(\omega t)$, then $\psi(e)$ is a periodic signal with Fourier series

$$u(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \sin(k\omega t) + b_k \cos(k\ \omega t))$$

$$a_k = \frac{2}{T}\int_a^b u(t)\cos(k\omega t)\, dt\, , b_k = \frac{2}{T}\int_a^b u(t)\sin(k\omega t)\, dt, a_0 = 0$$

- Then $y(t) = \sum_{k=1}^{\infty} G_k(a_k \sin(k\omega t + \phi_k) + b_k \cos(k\ \omega t + \phi_k))$ with $G_k = |G(j\omega k)|$ and $\phi_k = \arg(G(j\omega k))$

HARMONIC BALANCE

- Due to low-pass property of G(s), $y(t) = G_1(a_1\cos(\omega t + \phi_1) + b_1\sin(\omega t + \phi_1))$
- With $e(t) = A\, e^{j\omega t}$ we get $y(t) = G_1 e^{j\phi_1} e^{j\theta_1} e^{j\omega t} M_1$ and $M_1 = \sqrt{a_1^2 + b_1^2}, \theta_1 = \text{atan}\frac{a_1}{b_1}$
- For harmonic balance $\mathrm{e}(t) = -y(t)$, from which follows

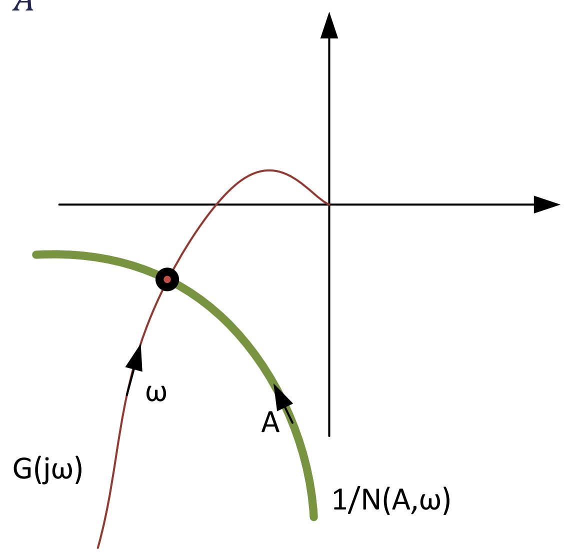
$$Ae^{j\omega t} = -G_1 e^{j\phi_1} e^{j\theta_1} e^{j\omega t} M_1 \Longrightarrow 1 + G_1 e^{j\theta_1} \frac{M_1}{A} e^{j\phi_1} = 0$$

- Harmonic balance equation

$$1 + G(j\omega)N(A,\omega) = 0$$

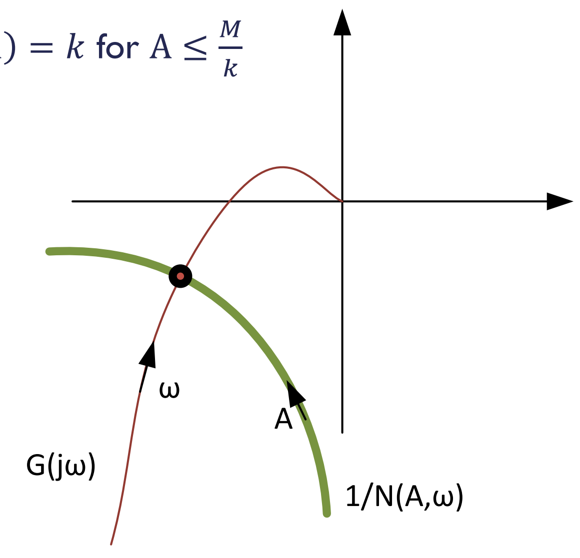
with $N(A,\omega) = \frac{1}{A}(b_1 + ja_1)$

- Graphical test using $1\,/N(A,\omega)$



DESCRIBING FUNCTIONS

- Signum function – Relay $N(A,\omega) = \frac{4M}{A\pi}$
- Saturation – $N(A) = \frac{2k}{\pi}\left[\arcsin\left(\frac{M}{kA}\right) + \left(\frac{M}{kA}\right)\sqrt{1-\left(\frac{M}{kA}\right)^2}\right]$ for $A > \frac{M}{k}$ or $N(A) = k$ for $\mathrm{A} \leq \frac{M}{k}$

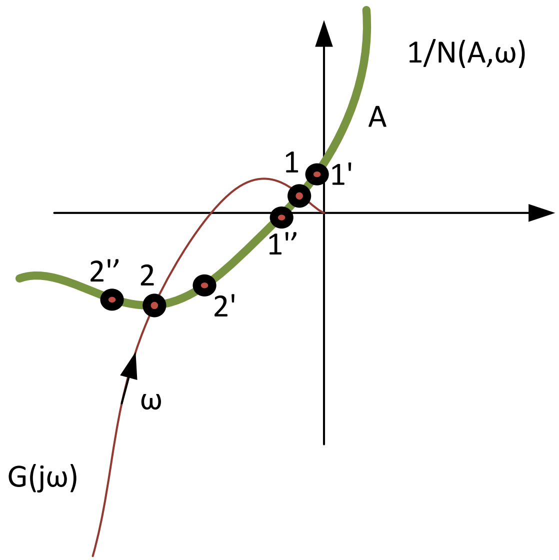


STABILITY OF LIMIT CYCLES

- Modified Nyquist criterion

$$1 + G(j\omega)N(A,\omega) = 0$$

- Points 1', 1'' can be viewed as critical points to $G(j\omega)$.
- Then apply Nyquist criterion.
- 1 is unstable: if 1 is perturbed to 1' (amplitude decreased), $G(j\omega)$ does not encircle 1' -> system is stable -> amplitude decreases further. If perturbed to 1'', system gets unstable and amplitude increases.
- 2 is stable: if 2 is perturbed to 2' (amplitude decreased), $G(j\omega)$ encircles 2' -> system is unstable -> amplitude increases and system gets back to 2. If perturbed to 2'', $G(j\omega)$ does not encircle 2'' -> system is stable -> amplitude decreases back to 2.



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