Time delay. LTV systems.

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Time-delay systems

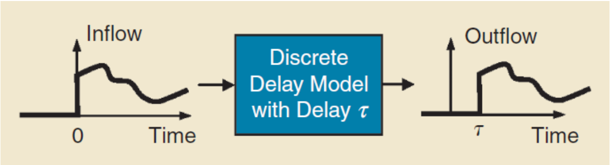
- transport delay in sensors / processing / controls (Moon/Mars rovers)
- transport delay in the plant / process (pipes, rolling mills, HVAC)

-time delay affects unfavourably closed loop stability (...phase-margin)
-time-delay is an infinite-dimensinal system (no finite state-space description ...)

Time-delay

time delay, dead time, transport delay

$$y(t) = u(t-\tau) \quad \Leftrightarrow \quad y(s) = u(s)e^{-\tau s}$$



- I/O time delay

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t-\tau)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

$$\det\left(s\mathbf{I} - \mathbf{A}\right)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t-\tau)$$

$$H(s, e^{-\tau s}) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}e^{-\tau s} = G(s)e^{-\tau s} = \frac{b(s)}{a(s)}e^{-\tau s}$$

- internal delay

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-\tau) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

$$\det\left(s\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_1 e^{-\tau s}\right)$$

$$H(s, e^{-\tau s}) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_1 e^{-\tau s}\right)^{-1}\mathbf{B} = \frac{b(s, e^{-\tau s})}{a(s, e^{-\tau s})}$$

CL stability: 1st order system with P controller

$$G(s) = \frac{b}{s+a} e^{-\tau s}$$

$$C(s) = k$$

$$T(s) = \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{\dfrac{kb}{s+a} e^{-\tau s}}{1 + \dfrac{kb}{s+a} e^{-\tau s}} = \frac{kbe^{-\tau s}}{s+a+kbe^{-\tau s}}$$

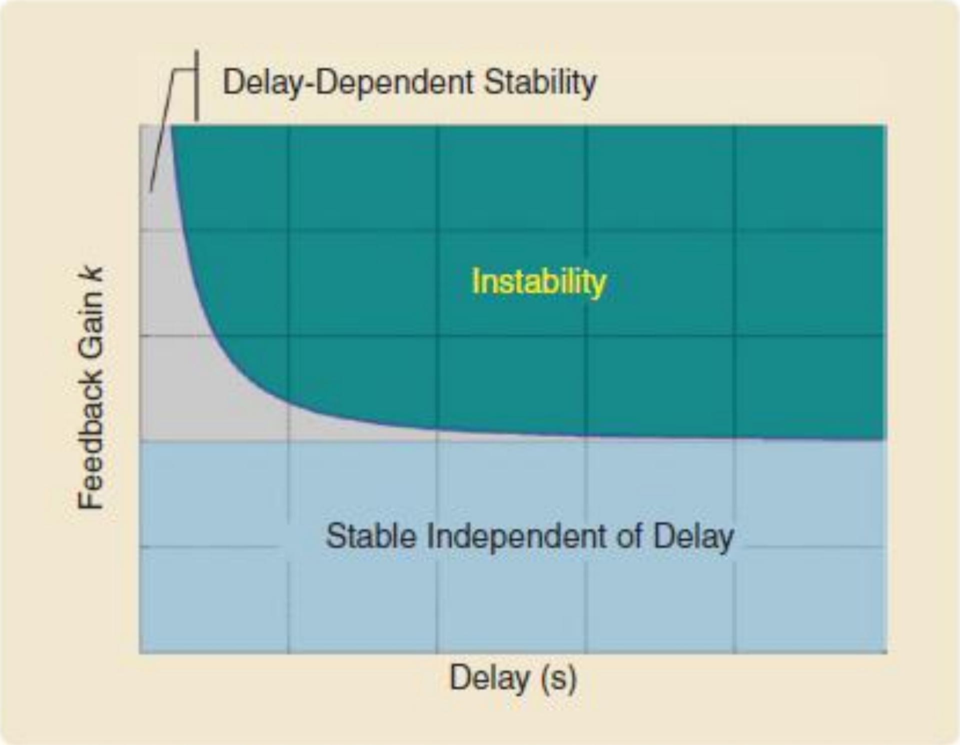


FIGURE 4 Stability chart. This chart is obtained for a closed-loop system with the plant transfer function $e^{-\tau s}b/(s+a)$ and the controller $C(s) = k$. This stability chart is partitioned into three regions, namely, delay-independent stable, delay-dependent stable, and unstable. This chart reveals the effect of a delay parameter on stability and how the controller gain k can be tuned to avoid instability.

Padé approximation

$$e^{-s} = 1 - s + \frac{s^2}{2!} - \frac{s^3}{3!} + \frac{s^4}{4!} - \cdots \quad \sim \quad \frac{b_1 s + b_0}{a_1 s + 1}$$

$$e^{-s} = 1 - s + \frac{s^2}{2!} - \frac{s^3}{3!} + \frac{s^4}{4!} - \cdots$$

$$\frac{b_1 s + b_0}{a_1 s + 1} = b_0 + (b_1 - b_0 a_1)s - a_1(b_1 - b_0 a_1)s^2 + \cdots$$

$$\Rightarrow \quad b_0 = 1, \quad (b_1 - b_0 a_1) = -1, \quad -a_1(b_1 - b_0 a_1) = \frac{1}{2}$$

$$b_0 = 1, \quad (b_1 - b_0 a_1) = -1, \quad -a_1(b_1 - b_0 a_1) = 1/2 \quad \Rightarrow \quad b_0 = 1, b_1 = -\frac{1}{2}, a_1 = \frac{1}{2} \quad \Rightarrow \quad e^{-s} \approx \frac{1 - s/2}{1 + s/2} \quad \Rightarrow \quad e^{-\tau s} \approx \frac{1 - \tau s/2}{1 + \tau s/2}$$

Pade approximation

$$e^{-s} \approx \frac{1 - s/2 + s^2/12}{1 + s/2 + s^2/12} \quad \Rightarrow \quad e^{-\tau s} \approx \frac{1 - \tau s/2 + (\tau s)^2/12}{1 + \tau s/2 + (\tau s)^2/12}$$

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pade(T,order)
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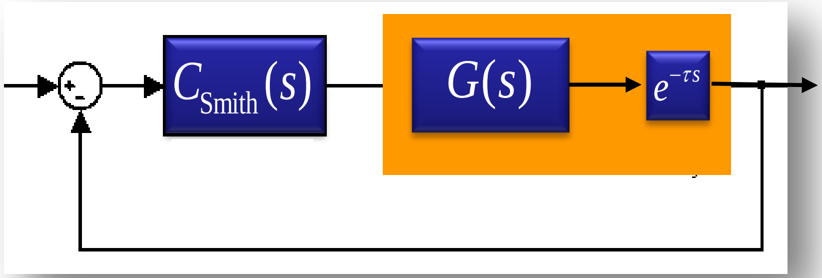
$$\tau \in (0,1) \quad \Rightarrow \quad e^{-\tau s} \approx \frac{1}{1 + \tau s/2}$$

Smith predictor

Goal (no delay in CL denominator):

$$T(s)=\frac{C(s)G(s)e^{-\tau s}}{1+C(s)G(s)}$$



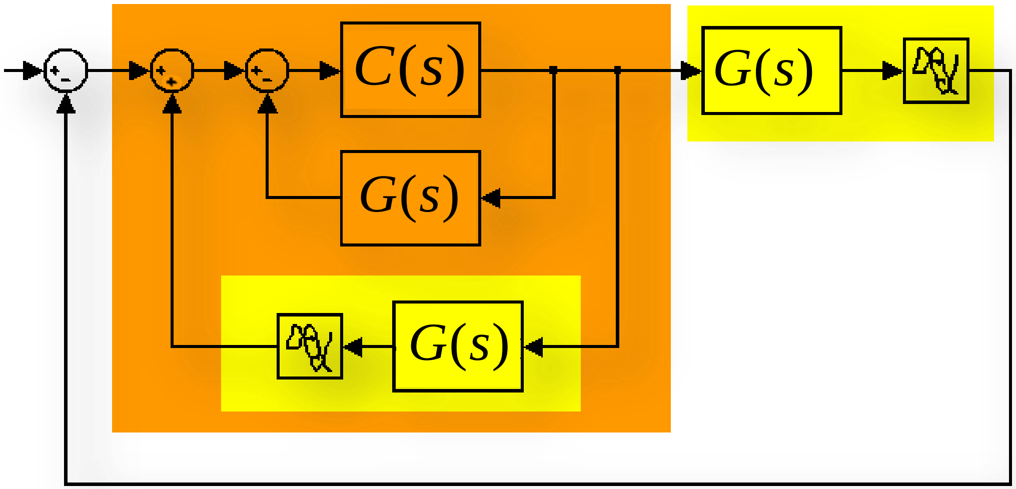
$$\frac{C(s)G(s)e^{-\tau s}}{1+C(s)G(s)}=T(s)=\frac{C_{\text{Smith}}(s)G(s)e^{-\tau s}}{1+C_{\text{Smith}}(s)G(s)e^{-\tau s}}$$

$$\boxed{C_{\text{Smith}}(s)=\frac{C(s)}{1+C(s)\left[G(s)-G(s)e^{-\tau s}\right]}}$$

$$T(s)=\frac{C(s)G(s)e^{-\tau s}}{1+C(s)G(s)}$$

Smithův prediktor

$$C_{\text{Smith}}(s) = \frac{C(s)}{1 + C(s)\left[G(s) - G(s)e^{-\tau s}\right]}$$



- sensitive w.r.t. delay estimate

Time-varying system stability (LTV)

$$A(t)=\begin{bmatrix} -1+1.5\cos^2 t & 1-1.5\sin t\cos t \\ -1-1.5\sin t\cos t & -1+1.5\sin^2 t \end{bmatrix}$$

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» syms t s
» A=[-1+1.5*cos(t)^2,1-1.5*sin(t)*cos(t);-1-
          1.5*sin(t)*cos(t),-1+1.5*sin(t)^2]
   A = [      -1+3/2*cos(t)^2,  1-3/2*sin(t)*cos(t)]
       [ -1-3/2*sin(t)*cos(t),      -1+3/2*sin(t)^2]

» eig(A)
   ans = [ -1/4+1/4*i*7^(1/2)]
         [ -1/4-1/4*i*7^(1/2)]
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$$\Phi(t,0)=\begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{bmatrix} \qquad x(t)=\Phi(t,0)x(0)$$