3 – Poles, zeros, responses

Martin Hromčík
Automatic Control

14-II-12
Transfer function poles

\[ g(s) = \frac{b(s)}{a(s)} \]

- subset of system poles

System poles

\[ c(s) = \det(sI - A) \]

- eigenvalues \( \lambda_i(A) \)
- internal dynamics, system’s resonances ...
- independent on B and C matrices (i.e. sensors and actuators). Compare to transfer function poles.

\[ s_i : a(s_i) = 0 \]
\[ g(s_i) = \infty \]

(mind zero-pole cancellation)
Transfer function zeros

\[ G(s) = \frac{b(s)}{a(s)} \]

\[ s_i : b(s_i) = 0 \]
\[ g(s_i) = 0 \]

(mind zero-pole cancellations)

- blocking zeros
- always input-output related
- non-minimum phase
- complications towards controls design (unstable poles/zeros)
- collocated / noncollocated control
System zeros

\[ q(s)/p(s) \quad p(s) = \det(sI - A) \]

\[ q(s) = C \text{adj}(sI - A)B + \det(sI - A)D = \]
\[ = \det(sI - A)[C(sI - A)^{-1}B + D] = \det \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} \]

- on top of the transfer zeros (no zeros-poles cancelled):
  - **input zeros** (=uncontrollable poles), \( z_i : \text{rank} \begin{bmatrix} z_iI - A & B \end{bmatrix} < n \)
  - **output zeros** (unobservable poles),

\[ z_i : \text{rank} \begin{bmatrix} z_iI - A \\ C \end{bmatrix} < n \]
Poles and zeros at infinity

\[ G(s) = s = \frac{s}{1} \quad \lim_{s \to \infty} G(s) = \infty \]

\[ G(s) = \frac{1}{s} \quad \lim_{s \to \infty} G(s) = 0 \]

- realizable systems / causal systems
- relative order
- \# poles = \#zeros (including those at infinity)
1st order system

Impulse response

\[ g(t) = ae^{-at} = \frac{1}{T} e^{-\frac{t}{T}} \]

\[ g(0^+) = -a^2 = -\frac{1}{T^2} \]

Step response

\[ h(t) = 1 - e^{-at} = 1 - e^{-\frac{t}{T}} \]

[Graphs showing impulse and step responses with time constants and rise and settling times indicated.]
2nd order system

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma - j\omega_d)(s + \sigma + j\omega_d)} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2} \]

(natural frequency) \[ \omega_n = \sqrt{b} \]

(exponential decay frequency) \[ \sigma = a/2 \]

(damping ratio) \[ \zeta = \frac{\sigma}{\omega_n} = \frac{1}{2\pi} \frac{T_n}{T_\sigma} = \frac{a/2}{\omega_n} = \cos \theta \]

(damped frequency) \[ \omega_d = \omega_n \sqrt{1 - \zeta^2} = b \sqrt{1 - (a/(2b))^2} \]

\[ G(s) = \frac{b}{s^2 + as + b} \quad a \geq 0 \]
\[ b > 0 \]
2\textsuperscript{nd} order system

Undamped system \( \sigma = 0, \omega_n = \omega_d, \zeta = 0 \)

\[
G(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}
\]

\[
g(t) = \omega_n \sin \omega_n t
\]

\[
h(t) = 1 - \cos \omega_n t
\]

Underdamped system \(|\zeta| < 1, \sigma = \zeta \omega_n, \omega_d = \omega_n \sqrt{1 - \zeta^2} \)

\[
G(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

\[
= \frac{\omega_n^2}{(s + \sigma - j\omega_d)(s + \sigma + j\omega_d)}
\]

\[
g(t) = \left( \frac{\omega_n^2}{\omega_d} \right) e^{-\sigma t} \sin \omega_d t
\]

\[
h(t) = 1 - e^{-\sigma t} \left[ \cos \omega_d t + \left( \frac{\sigma}{\omega_d} \right) \sin \omega_d t \right]
\]
2\textsuperscript{nd} order system

Critical damped system \( \zeta = 1, \sigma_{1,2} = \omega_n, \omega_d = 0 \)

\[
G(s) = \frac{\omega_n^2}{(s + \omega_n)^2}
\]

\[
g(t) = \omega_n^2 t e^{-\omega_n t}
\]

\[
h(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}
\]

Overdamped system \( |\zeta| \geq 1 \):

\[
G(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
= \frac{\omega_n^2}{(s + \sigma_1)(s + \sigma_2)}
\]

\[
\sigma_1 = \zeta \omega_n + \omega_n \sqrt{\zeta^2 - 1}
\]

\[
\sigma_2 = \zeta \omega_n - \omega_n \sqrt{\zeta^2 - 1}
\]

\[
g(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} (e^{-\sigma_2 t} - e^{-\sigma_1 t})
\]

\[
h(t) = \frac{\sigma_1 - \sigma_2 + \sigma_2 e^{-\sigma_1 t} - \sigma_1 e^{-\sigma_2 t}}{\sigma_1 - \sigma_2}
\]
2nd order system

<table>
<thead>
<tr>
<th>Poles</th>
<th>$\zeta$</th>
<th>Step response</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$-plane</td>
<td>0</td>
<td>Undamped</td>
</tr>
<tr>
<td>$\pm \zeta \omega_n \sqrt{1 - \zeta^2}$</td>
<td>$0 &lt; \zeta &lt; 1$</td>
<td>Underdamped</td>
</tr>
<tr>
<td>$s$-plane</td>
<td>$\zeta = 1$</td>
<td>Critically damped</td>
</tr>
<tr>
<td>$\pm \omega_n \sqrt{\zeta^2 - 1}$</td>
<td>$\zeta &gt; 1$</td>
<td>Overdamped</td>
</tr>
</tbody>
</table>

Undamped

Underdamped

Critically damped

Overdamped
Underdamped 2\textsuperscript{nd} order system

(settling time)

\[ T_s \doteq \frac{4}{\xi \omega_n} \]

(peek time)

\[ T_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}} \]

(overshoot)

\[ \%OS = 100e^{-\left(\frac{\xi \pi}{\sqrt{1-\xi^2}}\right)} \]

\[ \xi = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} \]

Rise time:

\[ T_r \approx \frac{1.8}{\omega_n} \]
Higher order systems. Dominant poles.

- 2\textsuperscript{nd} order approximation
- dominant poles (least damped)

\[ y(s) = \frac{bc}{s(s^2 + as + b)(s+c)} = \frac{A}{s} + \frac{Bs + C}{s^2 + as + b} + \frac{D}{s+c} \]

\[ D = \frac{-b}{c^2 + b - ca}, \quad C = \frac{ca - c^2}{c^2 + b - ca} \]

\[ A = 1, \quad B = \frac{ca - c^2}{c^2 + b - ca} \]

\[ c \to \infty: A = 1, B = -1, C = -a, D = 0 \]

- higher-order poles, more than 5x (10x) to the left, ignored