Exercises for lectures
14 – Modern Frequency Response Methods

Michael Šebek
Automatic control 2016
Feedback transfer functions

\[ y(s) = T(s)r(s) + S(s)d(s) - T(s)n(s) \]
\[ u(s) = K(s)S(s)r(s) - K(s)S(s)d(s) - K(s)S(s)n(s) \]
\[ e(s) = r(s) - y(s) = S(s)r(s) - S(s)d(s) + T(s)n(s) \]

- E.g. Influence of disturbance will be small for small S and influence of noise for small T. However, small S and small T is not possible at the same time.

**Proof**

\[ S(s) + T(s) = \frac{1}{1 + L(s)} + \frac{L(s)}{1 + L(s)} = \frac{1 + L(s)}{1 + L(s)} = 1 \]

- This is a serious limitation for the controller design. \( s = j\omega \), and for each individual \( \omega \)

\( S(j\omega) + T(j\omega) = 1 \)

- Therefore, design is a compromise: We have to choose priorities for individual frequency ranges.
- That is called **frequency response shaping** (loop shaping)
In the classic version of loop shaping it shapes OL characteristics, in spite the aim to shapes CL characteristics.

In certain frequency ranges, it is sufficient to shape $|L(j\omega)|$, therefore from the $|L(j\omega)|$ shape follows $|S(j\omega)| \text{ a } |T(j\omega)|$.

The precise relations:
\[
S(j\omega) = \frac{1}{1 + L(j\omega)} \quad T(j\omega) = \frac{L(j\omega)}{1 + L(j\omega)}
\]

The approximate relations:
- Typically for low frequency: $|L(j\omega)| \gg 1$  
  \[
  S(j\omega) \approx \frac{1}{L(j\omega)} \quad T(j\omega) = 1
  \]
- Typically for high frequency: $|L(j\omega)| \ll 1$  
  \[
  S(j\omega) = 1 \quad T(j\omega) = L(j\omega)
  \]

In the neighborhood of frequency $\omega_c$ (where $L$ is not low neither high) from the $|L(j\omega)|$ shape it doesn't follow shapes of $|S(j\omega)|$, $|T(j\omega)|$.

It also depends on the phase.
E.g. $|S(j\omega)|$, $|T(j\omega)|$ It can have a large peaks for $L(j\omega) \sim -1$
**PM, MS and MS Relations**

### Example

- **MS = 2**  \[\Rightarrow GM \geq 2, PM \geq 29^\circ\]

- **MT = 2**  \[\Rightarrow GM \geq 1.5, PM \geq 29^\circ\]

Therefore, it is easier to use MS or MT for specifications.

### Graphically:

**GM vs. MS**

\[ GM \geq 1 + \frac{1}{MT} \]

**GM vs. MT**

\[ GM \geq \frac{MS}{MS - 1} \]

**PM vs. MS and MT**

\[ PM \geq 2 \arcsin \left( \frac{1}{2MS} \right) \geq \frac{1}{MS} \text{ [rad]} \]

\[ PM \geq 2 \arcsin \left( \frac{1}{2MT} \right) \geq \frac{1}{MT} \text{ [rad]} \]
• unit FB should have a deviation less than 0.005 for all sinus signals with amplitude 1 and frequency below 100 Hz
• We can formulate these requirements by using the frequency weighting function:

Reference signals spectrum is $= 1$ for $0 \leq \omega \leq 200\pi$

because $e_b = 0.005$, the function is rectangle with height $1/0.005 = 200$
for a given frequency range
Example: Comparison with the classical req.

- The steady state error to step response
  \[ e_{\text{step,ss}} = \frac{1}{1 + \lim_{s \to 0} L(s)} = \lim_{s \to 0} S(s) = S(0) \]

- Classical requirement for steady state error to step response
  \[ |e_{\text{step,ss}}| \leq e_b \Rightarrow |e_{\text{step,ss}}| \frac{1}{e_b} \leq 1 \]

- It can be written as
  \[ |e_{\text{step,ss}}| \frac{1}{e_b} = |S(0)| \frac{1}{e_b} = |S(0)| W_1 \leq 1 \]

- This requirement is extend to frequency range as: \( \omega \in [0, \omega_1] \)
  \[ \forall \omega \in [0, \omega_1]: |S(j\omega)| W_1(\omega) \leq 1 \quad \Leftrightarrow \quad |S| W_1 \leq 1 \]

- \( W_1 \) is zero out of this range, the relation applies for all frequencies.

  \[ \forall \omega \in [0, \infty]: |S(j\omega)| W_1(\omega) \leq 1 \]
**Requirement to function** $L$

- **Requirement for control**
  \[ |S|W_1 \leq 1 \iff \forall \omega : |S(j\omega)|W_1(\omega) \leq 1 \]

- It can be also expressed with open loop transfer function
- For low frequencies

- approximately
  \[ |S(j\omega)| = \left| \frac{1}{1 + L(j\omega)} \right| \approx \left| \frac{1}{L(j\omega)} \right| \]

- \[ |S|W_1 \leq 1 \quad \text{or} \quad \frac{W_1}{|L|} \leq 1 \iff |L| \geq W_1 \]
- Or in more detail

\[ \forall \omega \in [0, \omega_0] : |L(j\omega)| \geq W_1(\omega) \]
Example: Uncertain system Nyquist graph

\[ G(j\omega) = G_0(j\omega)\left[1+W_2(\omega)\Delta(j\omega)\right], |\Delta(j\omega)| \leq 1, \]

\[ G_0(s) = \frac{2.5}{2.5s+1}, W_2(\omega) = \frac{4j\omega+0.2}{10j\omega+1} \]
Example: Uncertain system Nyquist and Bode graph

- System with multiplicative uncertainty
  \[ G(j\omega) = G_0(j\omega)[1 + W_2(\omega)\Delta(j\omega)], |\Delta(j\omega)| \leq 1 \]
  \[ G_0(s) = \frac{2.5}{(s+1)^3}, W_2(s) = 0.5 \]

- Nominal frequency response

- The total frequency characteristic
  \[ G(j\omega) = G_0(j\omega) \]
  \[ G_0(j\omega) \]

\[ g_0 = \frac{2.5}{(s+1)^3}; k = \text{rdf}(1); w = \text{rdf}(0.5); \omega = 0:0.01:2; \text{ball}(g_0, k, w, 1, j*\omega) \]
Consider system with transfer function

\[ G(s) = G_0(s) f(s) \]

where \( G_0(s) \) is given,

But \( f(s) \) is neglected and replace by multiplicative uncertainty.

Intensity of relative uncertainty is

\[
l_I(\omega) = \max_{G(s)} \left| \frac{G(j\omega) - G_0(j\omega)}{G_0(j\omega)} \right| = \max_{f(s)} |f(j\omega) - 1| \]

We discuss two cases:

- Time delay neglecting

\[ f(s) = e^{-\theta s}, 0 \leq \theta \leq \theta_{\text{max}} \]

- First order term neglecting

\[ f(s) = \frac{1}{(\tau_p s + 1)}, 0 \leq \tau_p \leq \tau_{\text{max}} \]
Time delay neglecting

- Consider $G(s) = G_0(s)f(s)$, where $f(s) = e^{-\theta s}, 0 \leq \theta \leq \theta_{\text{max}}$
- For maximum delay is variance $l_1(\omega) = |e^{-j\omega \theta_{\text{max}}} - 1|$ depicted in the picture (for $\theta_{\text{max}} = 2$)
- reaches 1 for $\omega = 1/\theta_{\text{max}}$
- maximum (=2) for $\omega = \pi/\theta_{\text{max}}$
- Then oscillates between 0 and 2
- It is similar for other $\theta$

$$l_1(\omega) = \begin{cases} |e^{-j\omega \theta_{\text{max}}} - 1|, & \omega < \pi/\theta_{\text{max}} \\ 2, & \omega \geq \pi/\theta_{\text{max}} \end{cases}$$

- Rational function replacement 1st order and 3rd order
Time delay neglecting

\[ W_{2,2}(j\omega) = W_{2,1}(j\omega) \left| \frac{s^2 + 2.1s + 1}{s^2 + 1.4s + 1} \right|_{s = j\omega} \]

\[ W_{2,1}(j\omega) = \left| \frac{4s + 0.2}{10s + 1} \right|_{s = j\omega} \]

\[ \frac{k}{\tau s + 1} e^{-\theta s} \left| \frac{1}{G_0(s)} - 1 \right| \]

\( k, \tau, \theta \in \{2, 2.5, 3\} \)
Consider $G(s) = G_0(s)f(s)$, with $f(s) = 1/(\tau_p s + 1), 0 \leq \tau_p \leq \tau_{\text{max}}$

- variation
  
  $l_1(\omega) = \left|1 - \frac{1}{(\tau_{\text{max}} s + 1)}\right|$

- From figure
- for $\tau_{\text{max}} = 2$ red
- and for smaller $\tau$ blue

- it is represented by
  
  a rational weighting function

\[
|w_1(j\omega)| = l_1(\omega)
\]

\[
w_1(j\omega) = 1 - \frac{1}{\tau_{\text{max}} s + 1} = \frac{\tau_{\text{max}} s}{\tau_{\text{max}} s + 1}
\]
Robust stability conditions - proof

- Consider nominal CL system stability. Nyquist graph $L_0(s)=D(s)G_0(s)$ It therefore meets the Nyquist stability criterion.
- Furthermore CL system doesn't have a pole on the imaginary axis $1+L_0(s)$ doesn't have a zero on the imaginary axis $1+L_0(j\omega) \neq 0$, $\forall \omega$
- For robust stability it must also apply for $1+L(s)$ for each $\omega$ and $\Delta$

\[
1 + L(j\omega) \neq 0 \; \forall \omega, \; \forall |\Delta(j\omega)| \leq 1
\]

\[
1 + L_0(j\omega) + L_0(j\omega)W_2(\omega)\Delta(j\omega) \neq 0 \; \forall \omega, \; \forall |\Delta(j\omega)| \leq 1
\]

\[
1 + L_0(j\omega) + \frac{1+L_0(j\omega)}{1+L_0(j\omega)} L_0(j\omega)W_2(\omega)\Delta(j\omega) \neq 0 \; \forall \omega, \; \forall |\Delta(j\omega)| \leq 1
\]

nominal

\[
(1+L_0(j\omega))(1+T_0(j\omega)W_2(\omega)\Delta(j\omega)) \neq 0 \; \forall \omega, \; \forall |\Delta(j\omega)| \leq 1
\]


\[
(1+T_0(j\omega)W_2(\omega)\Delta(j\omega)) \neq 0 \; \forall \omega, \; \forall |\Delta(j\omega)| \leq 1
\]

\[
T_0(j\omega)W_2(\omega)\Delta(j\omega) \neq 1 \; \forall \omega, \; \forall |\Delta(j\omega)| \leq 1
\]

\[
|T_0(j\omega)W_2(\omega)| < 1 \; \forall \omega
\]
• European Southern Observatory: four 8 m telescopes (~16 m), Atacama in Chile
• Precise targeting and disturbance (wind) suppression with robust control methods
• Department project, Z. Hurák: AUTOMA 1/05
• video
The first low damped structure mode

Wind gusts

Model order: 60 (the finite element method)

Model order reduction

Reduced model of order: 24

Example: VLT ESO
Active and adaptive optics
• plan 2015:
• OWL Telescope, mirror 100m,
• deformable segments
• 500,000 actuators
• design?
• Numerical methods?
• VIDEO OWL.mpeg
Example: Waterbed effect I

- For \( L(j\omega) = \frac{4}{(s+2)(s-1)} \) with one unstable pole.

- is \( S \) stable \( S(j\omega) = \frac{-2 + s + s^2}{2 + s + s^2} \)

- however

\[
\int_0^\infty \ln|S(j\omega)|d\omega = \pi
\]

- from Bode diagram

\[
|S(j\omega)| > 1 \ \forall \omega
\]

- It is understandable because we must pay something for stabilization.
Relative order condition can be omitted
Then general relationship applies

\[
\int_0^\infty \ln |S(j\omega)| \, d\omega = \pi \sum_{i=0}^{n_p} \Re p_{unstable,i} - \frac{\pi}{2} \lim_{s \to \infty} sL(s)
\]
• let's compare the non-minimum phase transfer function

\[ L(j\omega) = \frac{1}{1 + s} \frac{1 - s}{1 + s} \]

• With its minimum phase "counterpart"

\[ L_m(j\omega) = \frac{1}{1 + s} \]

• Phase lag (unstable zero) shift the graph to the red circle.
Example: Waterbed effect II.

- Consider the non-minimal phase system \( G(s) = \frac{2-s}{2+s} \) and controller \( \frac{k}{s} \)

\[
L(s) = \frac{k}{s} \frac{2-s}{2+s}
\]

- Sensitivity S Bode diagram for \( k = 0.1, 0.5, 1.0 \) and 2.0

- With increasing gain also increase influence of unstable zero (and sensitivity peak)

- System becomes unstable for \( k = 2 \)