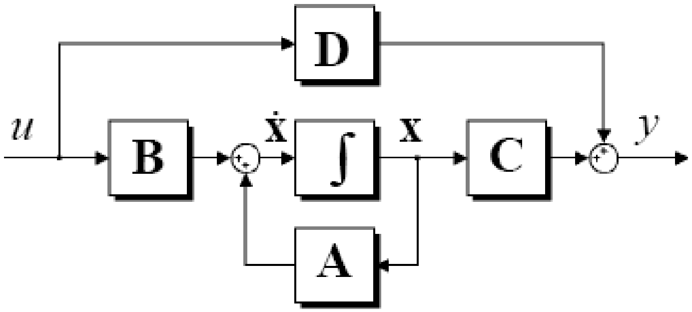
Continuous-time models. State-space.

- LTI state-space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \qquad \mathbf{x}(0^-) = \mathbf{x}_0$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$



$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + [d] u$$

- Time-domain solution (u and x0 given)

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$
$$= \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}u(\tau)d\tau$$

$\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ ... state transition matrix (suggest computation methods ...)

Continuous-time models. State-space.

- Frequency-domain (Laplace transform) solution

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \qquad \mathbf{x}(0^-) = \mathbf{x}_0$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

A	**B**
C	**D**

$$\mathcal{L}_-(\dot{\mathbf{x}}) = s\mathbf{x}(s) - \mathbf{x}(0^-) = s\mathbf{x}(s) - \mathbf{x}_0$$

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0$$
$$\mathbf{y}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{u}(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0$$

$$y(s) = \frac{b(s)}{a(s)}u(s) + \frac{c_{x_0}(s)}{a(s)}$$

$$\mathbf{x}(t) = \mathcal{L}_-^{-1}\{\mathbf{x}(s)\},\ \mathbf{y}(t) = \mathcal{L}_-^{-1}\{\mathbf{y}(s)\}$$

Continuous-time models. State-space.

- Transfer function *G*(*s*)

$$y(s) = \boxed{\frac{b(s)}{a(s)}}\, u(s) + \frac{r_{x_0}(s)}{a(s)}$$

$$G(s) = \frac{b(s)}{a(s)}$$

$$a(s) = \det(s\mathbf{I} - \mathbf{A})$$
$$b(s) = \mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + a(s)\mathbf{D}$$

- explain poles and zeros (SISO)
- mind zero-pole cancellation
- relative order
- characteristic polynomial
- eigenvalues of A = poles of G(s)

- Related differential equation

$$a_n y^{(n)}(t) + \cdots + a_1 \dot{y} + a_0 y(t) = b_n u^{(n)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

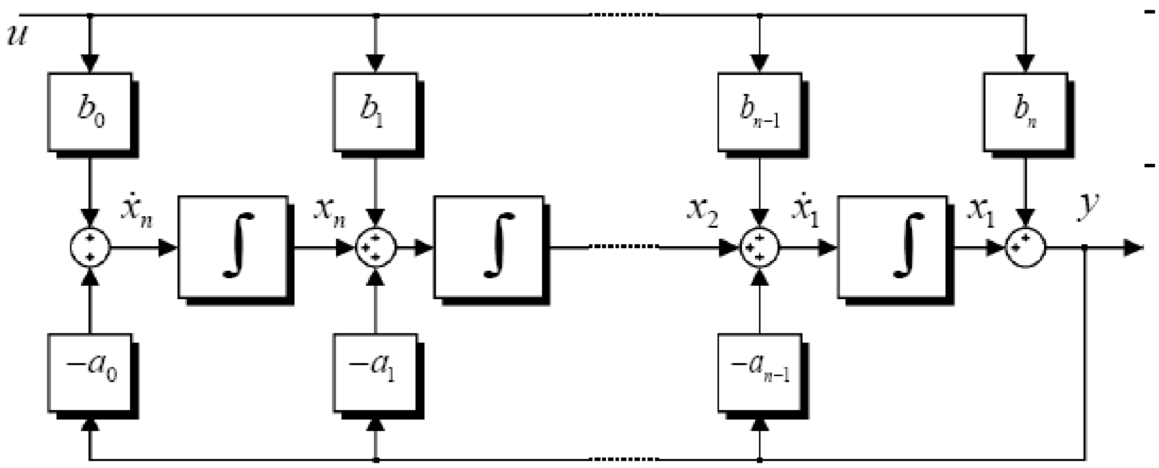
Continuous-time models. I/O.

- Now ... the other way round ☺. Starting with the I/O model (differential equation)

$$a_n y^{(n)}(t) + \cdots + a_1 \dot{y} + a_0 y(t) = b_n u^{(n)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

$$y^{(n-1)}(0^-), \ldots, \dot{y}(0^-), y(0^-)$$

- Related block diagram



- note the indicated state-space equivalent model of the given IO model
- one particular realization, see further for more ...

- Laplace transform, transfer function

$$\mathcal{L}_-\left\{y^{(k)}\right\} = s^k y(s) - s^{k-1} y(0^-) - \cdots - y^{(k-1)}(0^-)$$

$$\Downarrow$$

$$\left(a_n s^n + \cdots + a_1 s + a_0\right) y(s) = \left(b_n s^n + \cdots + b_1 s + b_0\right) u(s) + c(s) \Rightarrow$$

$$y(s) = \frac{b(s)}{a(s)} u(s) + \frac{c(s)}{a(s)}$$

Continuous-time models. I/O.

- State-space realization

given

$$a_n y^{(n)}(t) + \cdots + a_1 \dot{y} + a_0 y(t) = b_n u^{(n)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

find equivalent

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \qquad \mathbf{x}(0^-) = \mathbf{x}_0$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$

- Principal questions

existence (mind LTI systems with non-rational transfer)
uniqueness

orders and minimal realizations
special structures

Continuous-time models. I/O.

- State-space special realizations

Frobenius canonical controllability forms (all for normalized *a*(*s*), *an*=1)

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & & 0\\ \vdots & & & \ddots & 0\\ 0 & 0 & 0 & & 1\\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1}\end{bmatrix},\mathbf{B}=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}\bar{b}_0 & \bar{b}_1 & \dots & \bar{b}_{n-1}\end{bmatrix}, \qquad \mathbf{D}=\begin{bmatrix}b_n\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}-a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0\\ 1 & 0 & & 0 & 0\\ 0 & 1 & & 0 & 0\\ \vdots & & \ddots & & \\ 0 & 0 & & 1 & 0\end{bmatrix},\mathbf{B}=\begin{bmatrix}1\\0\\\vdots\\0\\0\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}\bar{b}_{n-1} & \dots & \bar{b}_1 & \bar{b}_0\end{bmatrix}, \qquad \mathbf{D}=\begin{bmatrix}b_n\end{bmatrix}$$

Jordan canonical form

the *A* matrix in Jordan form, diagonal or block-diagonal
achieved by partial fraction expansion of the rational transfer function

Continuous-time models. I/O.

- State-space special realizations

canonical observability forms

$$\mathbf{A} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & & 0 \\ \vdots & & & \ddots & 0 \\ -a_1 & 0 & 0 & & 1 \\ -a_0 & 0 & 0 & \dots & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{n-1} - a_{n-1}b_n \\ b_{n-2} - a_{n-2}b_n \\ \vdots \\ b_1 - a_1 b_n \\ b_0 - a_0 b_n \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} b_n \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \dots & 0 & 0 & -a_0 \\ 1 & & 0 & 0 & -a_1 \\ & \ddots & & & \\ 0 & & 1 & 0 & -a_{n-2} \\ 0 & & 0 & 1 & -a_{n-1} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_0 - a_0 b_n \\ b_1 - a_1 b_n \\ \vdots \\ b_{n-2} - a_{n-2}b_n \\ b_{n-1} - a_{n-1}b_n \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} b_n \end{bmatrix}$$

- State-space realizations equivalence (for a given system)

$$\mathbf{A}_{\text{new}} = \mathbf{T}^{-1}\mathbf{A}_{\text{old}}\mathbf{T}$$
$$\mathbf{B}_{\text{new}} = \mathbf{T}^{-1}\mathbf{B}_{\text{old}}$$
$$\mathbf{C}_{\text{new}} = \mathbf{C}_{\text{old}}\mathbf{T}$$

- transfer function not affected by the transform
- linear transform of coordinates (as known from linear algebra basics):

$$\mathbf{x}_{\text{new}} = \mathbf{V}\mathbf{x}_{\text{old}} \qquad \mathbf{T} = \mathbf{V}^{-1}$$

Continuous-time models. System modes.

- partial fraction expansion of .../*a*(*s*)

$$y(s)=\frac{[\mathbf{C}\,\mathrm{adj}(s\mathbf{I}-\mathbf{A})\mathbf{B}+\mathbf{D}]}{\det(s\mathbf{I}-\mathbf{A})}u(s)+\frac{\mathbf{C}\,\mathrm{adj}(s\mathbf{I}-\mathbf{A})\mathbf{x}_0}{\det(s\mathbf{I}-\mathbf{A})}=\frac{b(s)}{a(s)}\frac{n_u(s)}{d_u(s)}+\frac{c_{\mathbf{x}_0}(s)}{a(s)}$$

$$a(s)=\det(sI-A)$$
$$=\prod(s-a_i)\prod(s-b_j)^{m_j}\prod\left((s+\sigma_k)^2+\omega_k^2\right)\prod\left((s+\sigma_l)^2+\omega_l^2\right)^{n_l}$$

- related time respones = modes

$$e^{a_i t}\quad e^{b_j t},te^{b_j t},\ldots,t^{m_j-1}e^{b_j t}/(m_j-1)!\quad e^{-\sigma t}\sin\omega_k t,e^{-\sigma t}\cos\omega_k t\quad \ldots,te^{-\sigma t}\cos\omega_l t,te^{-\sigma t}\sin\omega_l t,\ldots$$

- example / experiment ☺

oscillatory modes excited by impulse inputs or disturbance, or by initial condition