5 – Systems interconnections.

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Interconnections

$$y_i(s) = F_i(s)u_i(s) \qquad F_i(s) = \frac{b_i(s)}{a_i(s)}$$

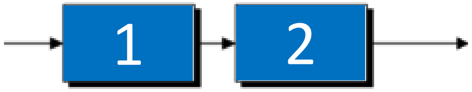
$$\dot{\mathbf{x}}_i = \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i u_i$$
$$y_i = \mathbf{C}_i\mathbf{x}_i + \mathbf{D}_i u_i$$

$$a_i(s) = \det\left(s\mathbf{I} - \mathbf{A}_i\right) \qquad i = 1,2$$

$$y(s) = F(s)u(s), \quad F(s) = \frac{b(s)}{a(s)}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u \qquad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$
$$a(s) = \det\left(s\mathbf{I} - \mathbf{A}\right)$$

Serial connectioon (cascade)



$$y(s)=y_2(s)=F_2(s)u_2(s)=F_2(s)y_1(s)=F_2(s)F_1(s)u_1(s)=F_2(s)F_1(s)u(s)$$

$$y(s)=F(s)u(s),F(s)=F_2(s)F_1(s)$$

$$\frac{b(s)}{a(s)}=\frac{b_2(s)b_1(s)}{a_2(s)a_1(s)}$$

$$\dot{\mathbf{x}}_1=\mathbf{A}_1\mathbf{x}_1+\mathbf{B}_1u_1=\mathbf{A}_1\mathbf{x}_1+\mathbf{B}_1u$$

$$\dot{\mathbf{x}}_2=\mathbf{A}_2\mathbf{x}_2+\mathbf{B}_2u_2=\mathbf{A}_2\mathbf{x}_2+\mathbf{B}_2\left(\mathbf{C}_1\mathbf{x}_1+\mathbf{D}_1u\right)$$

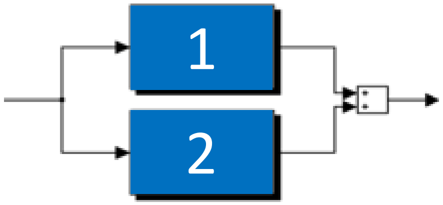
$$y=y_2=\mathbf{C}_2\mathbf{x}_2+\mathbf{D}_2u_2=\mathbf{C}_2\mathbf{x}_2+\mathbf{D}_2\mathbf{C}_1\mathbf{x}_1+\mathbf{D}_2\mathbf{D}_1u$$

$$\dot{\mathbf{x}}=\begin{bmatrix}\mathbf{A}_1 & \mathbf{0}\\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2\end{bmatrix}\mathbf{x}+\begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\mathbf{D}_1\end{bmatrix}u$$

$$y=\begin{bmatrix}\mathbf{D}_2\mathbf{C}_1 & \mathbf{C}_2\end{bmatrix}\mathbf{x}+\mathbf{D}_2\mathbf{D}_1u$$

$$a(s)=a_1(s)a_2(s)=\det(s\mathbf{I}-\mathbf{A})=\det(s\mathbf{I}-\mathbf{A}_1)\det(s\mathbf{I}-\mathbf{A}_2)$$

Paralel connection



$$y(s) = y_1(s) + y_2(s) = \left(F_1(s) + F_2(s)\right)u(s)$$

$$y(s) = F(s)u(s), F(s) = F_1(s) + F_2(s)$$

$$\frac{b(s)}{a(s)} = \frac{b_1(s)}{a_1(s)} + \frac{b_2(s)}{a_2(s)} = \frac{a_2(s)b_1(s) + a_1(s)b_2(s)}{a_1(s)a_2(s)}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}u$$

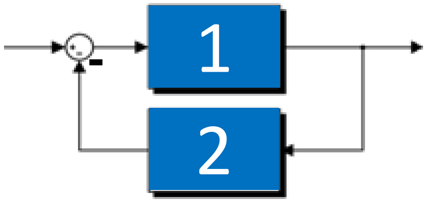
$$y = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}\mathbf{x} + \left(\mathbf{D}_2 + \mathbf{D}_1\right)u$$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

$$a(s) = a_1(s)a_2(s) = \det(sI - A) = \det(sI - A_1)\det(sI - A_2)$$

Feedback(-s)

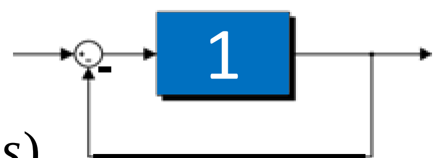
$$u_1 = u - y_2$$
$$y_1 = u_2 = y$$



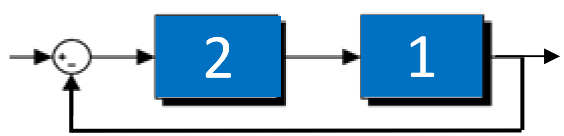
$$y(s) = F_1(s)\big(u(s) - y_2(s)\big)$$
$$= F_1(s)\big(u(s) - F_2(s)u_2(s)\big)$$
$$= F_1(s)u(s) - F_1(s)F_2(s)y(s)$$
$$\big(1 + F_1(s)F_2(s)\big)\, y(s) = F_1(s)u(s)$$
$$y(s) = \frac{F_1(s)}{1 + F_1(s)F_2(s)} u(s)$$

$$f(s) = \frac{F_1(s)}{1 + F_1(s)F_2(s)}$$
$$\frac{b(s)}{a(s)} = \frac{a_2(s)b_1(s)}{a_1(s)a_2(s) + b_1(s)b_2(s)}$$

$$F_2(s) = 1$$



$$F(s) = \frac{F_1(s)}{1 + F_1(s)}$$
$$\frac{b(s)}{a(s)} = \frac{b_1(s)}{a_1(s) + b_1(s)}$$



$$f_1(s) \,\square\, f_1(s) f_2(s)$$

$$F(s) = \frac{F_1(s)F_2(s)}{1 + F_1(s)F_2(s)}$$
$$\frac{b(s)}{a(s)} = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s) + b_1(s)b_2(s)}$$

Feedback(-s)

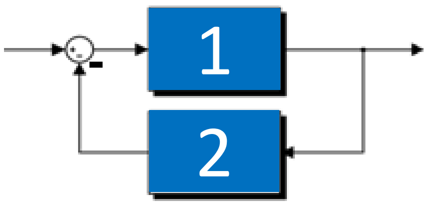
$$\mathbf{D}_1 = \mathbf{0}, \mathbf{D}_2 = \mathbf{0}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1\mathbf{C}_2 \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}\mathbf{u}$$

$$y = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \end{bmatrix}\mathbf{x}$$

$$u_1 = u - y_2$$

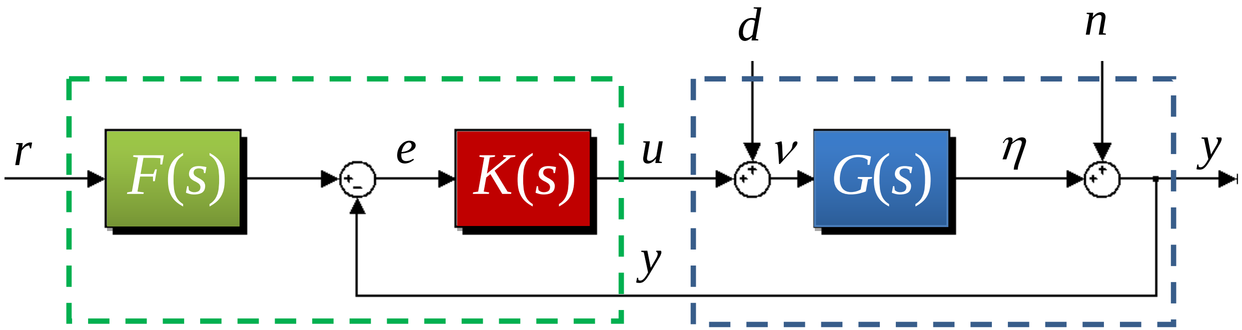
$$y_1 = u_2 = y$$



$$c(s) = \det(s\mathbf{I} - \mathbf{A}) = \det(s\mathbf{I} - \mathbf{A}_1)\det(s\mathbf{I} - \mathbf{A}_2)\det\left(\mathbf{I} - \underbrace{\mathbf{C}_2(s\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{B}_2}_{F_2(s)}\underbrace{\mathbf{C}_1(s\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{B}_1}_{F_1(s)}\right)$$

2 DoF controller

$$y = G\left(u + d\right) + n$$
$$u = K(Fr - y)$$



$$y = \frac{GKF}{1+GK} r + \frac{G}{1+GK} d + \frac{1}{1+GK} n$$
$$\eta = \frac{GKF}{1+GK} r + \frac{G}{1+GK} d - \frac{GK}{1+GK} n$$
$$v = \frac{KF}{1+GK} r + \frac{1}{1+GK} d - \frac{K}{1+GK} n$$
$$u = \frac{KF}{1+GK} r - \frac{GK}{1+GK} d - \frac{K}{1+GK} n$$
$$e = \frac{F}{1+GK} r - \frac{G}{1+GK} d - \frac{1}{1+GK} n$$

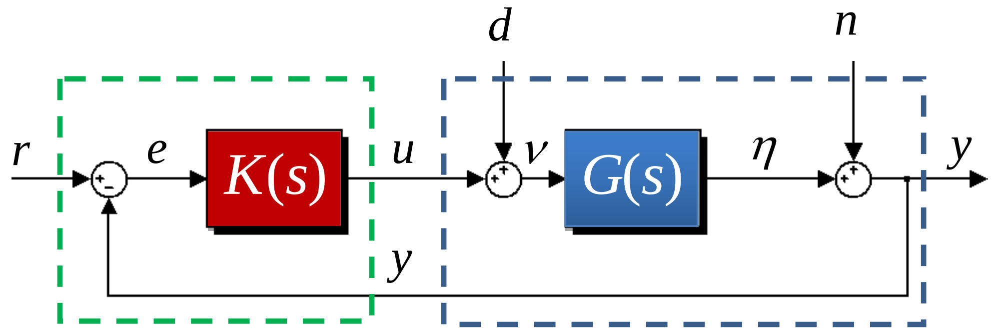
$$\varepsilon = r - \eta = \left(1 - \frac{GKF}{1+GK}\right) r - \frac{G}{1+GK} d + \frac{GK}{1+GK} n$$

„Gang of Six“

$$TF = \frac{GKF}{1+GK}, \quad T = \frac{GK}{1+GK}, \quad GS = \frac{G}{1+GK}$$
$$KFS = \frac{KF}{1+GK}, \quad KS = \frac{K}{1+GK}, \quad S = \frac{1}{1+GK}$$

1 DoF controller

$$y = G\left(u + d\right) + n \qquad F = 1$$
$$u = K(r - y)$$



$$y = \frac{GK}{1+GK} r + \frac{G}{1+GK} d + \frac{1}{1+GK} n$$
$$\eta = \frac{GK}{1+GK} r + \frac{G}{1+GK} d - \frac{GK}{1+GK} n$$
$$v = \frac{K}{1+GK} r + \frac{1}{1+GK} d - \frac{K}{1+GK} n$$
$$u = \frac{K}{1+GK} r - \frac{GK}{1+GK} d - \frac{K}{1+GK} n$$
$$e = \frac{1}{1+GK} r - \frac{G}{1+GK} d - \frac{1}{1+GK} n$$

$$\varepsilon = r - \eta = \left(1 - \frac{GK}{1+GK}\right) r - \frac{G}{1+GK} d + \frac{GK}{1+GK} n$$

„Gang of Four“

$$T = \frac{GK}{1+GK}, \quad GS = \frac{G}{1+GK}$$
$$KS = \frac{K}{1+GK}, \quad S = \frac{1}{1+GK}$$

Struktura s jedním regulátorem

- Open loop

$$L = GK$$

- Sensitivity (response to dist.)

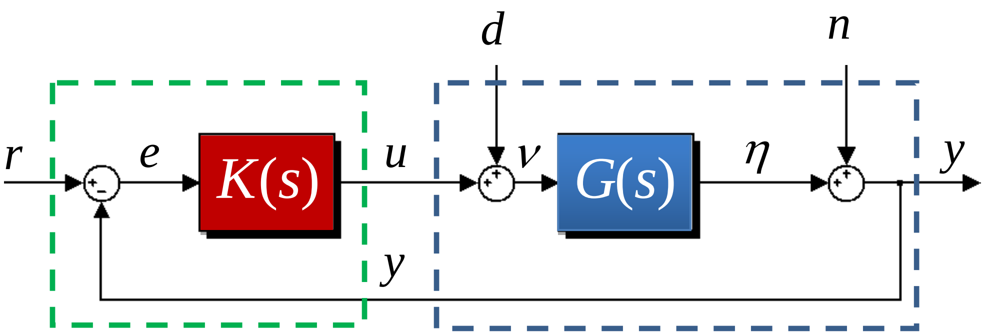
$$S = \frac{1}{1+GK}$$

- Complementary sensiitivity (CL transfer function)

$$T = \frac{GK}{1+GK}$$

- Input sensitivity $GS = \dfrac{G}{1+GK}$

- Output sensitivity $KS = \dfrac{K}{1+GK}$



$$y = Tr + GSd + Sn$$ output

$$\eta = Tr + GSd - Tn$$

$$u = KSr - Td - KSn$$ control input

$$e = Sr - GSd - Sn$$

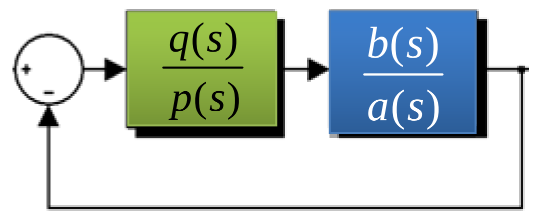
$$\varepsilon = Sr - GSd + Tn$$ control error

Closed loop dynamics

$$G(s)=\frac{b(s)}{a(s)},\quad K(s)=\frac{q(s)}{p(s)} \qquad L(s)=G(s)K(s)=\frac{b(s)q(s)}{a(s)p(s)}$$

$$S(s)=\frac{1}{1+L(s)}=\frac{a(s)p(s)}{a(s)p(s)+b(s)q(s)}$$

$$T(s)=\frac{L(s)}{1+L(s)}=\frac{b(s)q(s)}{a(s)p(s)+b(s)q(s)}$$



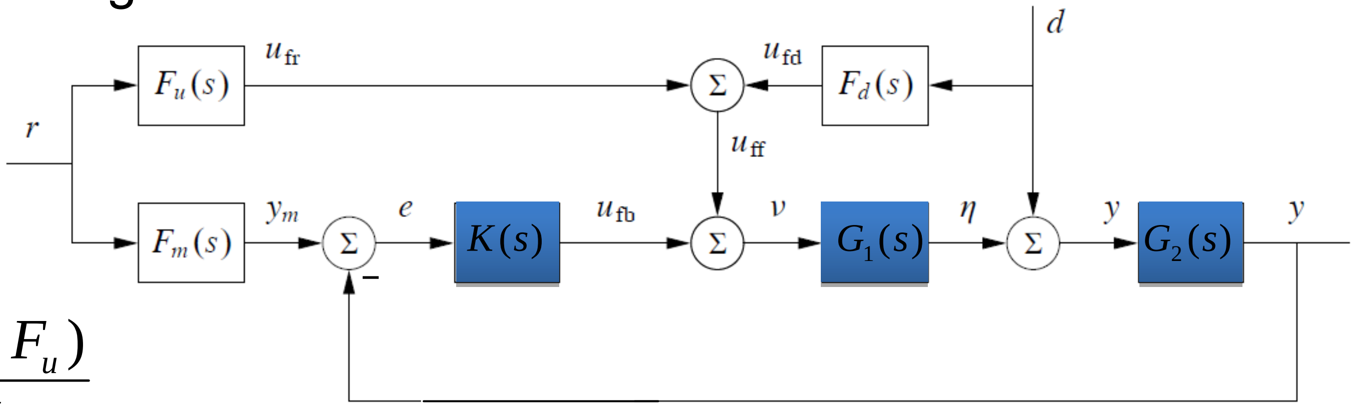
$$c(s)=a(s)p(s)+b(s)q(s)$$

Observations:

- unstable poles resp. zeros of L always appear as unstable zeros of S and T respectively (can never be cancelled). Leading to non-minimum phase CL system.
- unstable poles cannot be eliminated (while unstable poles can be stabilized by FB)

Exercise (HW): further interconnections

- Devise the following formulas:



$$T_{ry} = \frac{G_2(KF_m + F_u)}{1+GK}$$

$$T_{ry} = \frac{G(1+F_dG_1)}{1+GK} = F_m + \frac{GF_u - F_m}{1+GK}, G = G_1G_2$$

$$F_u \approx \frac{F_m}{G}, F_d \approx -\frac{1}{G}$$

- Similarly for: