Polynomials – properties. A review.

- Ring of polynomials (not field).
- Therefore: divisor, least common divisor, coprime polynomials

\[ g(s) = \gcd(a(s), b(s)) \Leftrightarrow \exists p(s), q(s), r(s), v(s): \]

\[ a(s)p(s) + b(s)q(s) = g(s) \]

\[ a(s)v(s) + b(s)w(s) = 0 \]

\[ U(s) = \begin{bmatrix} p(s) & q(s) \\ v(s) & w(s) \end{bmatrix}, \det U(s) \in \mathbb{R} \]

- Division with remainder – the Euclid’s algorithm:

\[ a(s) = b(s)q(s) + r(s), \ \deg r(s) < \deg b(s) \]

- Linear equations \( a(s)x(s) = b(s) \) do not have solutions typically since

\[ x(s) = b(s)/a(s) \] is not generically a polynomial

- Diophantine equations:

\[ a(s)x(s) + b(s)y(s) = c(s) \]
Properties of polynomial equations

Polynomial Diophantine equation

\[ a(s)x(s) + b(s)y(s) = c(s) \]

\[ g = \gcd(a, b) \]

\[ \overline{a} = a/g \]

\[ \overline{b} = b/g \]

• Solution exists iff \( g \mid c \)

All solutions:

\[ x(s) = x'(s) - \overline{b}(s)t(s) \]

\[ y(s) = y'(s) + \overline{a}(s)t(s) \]

\( t(s) \) is a free polynomial parameter

Minimum degree solutions:

• There exists a unique solution such that \( \deg x < \deg \overline{b} \)
  i.e. minimal degree in \( x \)

• There exists a unique solution such that \( \deg y < \deg \overline{a} \)
  i.e. minimal degree in \( x \)

• These are the same if \( \deg c < \deg a + \deg b \), otherwise they differ

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1. Select desirable CL poles. Compose the closed-loop characteristic polynomial. Solve $a(s)x(s) + b(s)y(s) = c(s)$

2. Select appropriate solution.

**Case 1:** $a(s)$, $b(s)$ coprime
- no hidden modes
- $c(s)$ can be an arbitrary polynomial (of sufficient order / degree though – otherwise, non-proper controller $y/x$)

**Case 2:** $a(s)$, $b(s)$ not coprime: $\gcd(a(s), b(s)) = g(s)$
- hidden modes
  $$g(s)\overline{a}(s)x(s) + g(s)\overline{b}(s)y(s) = g(s)\overline{c}(s)$$
- $c(s)$ must contain $g(s)$
- hidden modes cannot be re-assigned (modified)
- other poles can be placed arbitrarily
Pole placement: proper controllers

In order to get a proper controller:
• if $G(s)$ is strictly proper, $\deg a(s) = n$, then select
  1) $c(s)$ of degree at least $2n-1$ and
  2) select minimal-solution (minimal degree of $y$), tedy
• Tím je zaručeno, že vyjde ryzí regulátor řádu $n-1$

Explanation / “proof”:

\[
\begin{align*}
  a(s)x(s) + b(s)y(s) &= c(s) \\
  \deg x(s) &= n - 1
\end{align*}
\]
All stabilizing controllers

Assures closed-loop stability.

All stabilizing controllers for a given system $G(s) = b(s)/a(s)$, with $a(s)$, $b(s)$, are given as $C(s) = q(s)/p(s)$ where $p(s)$, $q(s)$ solve the equation

$$a(s)p(s) + b(s)q(s) = c(s)$$

for all stable polynomials $c(s)$

Conditions:
$G(s)$ has no hidden unstable modes, i.e. $\gcd(a,b)$ must be stable

Proper controller
For proper system, select $\deg c(s) \geq 2\deg a(s) - 1$
and select minimum-degree-of $q(s)$ solution
Model matching

• make the closed loop transfer function equal to a prescribed one:

\[
\frac{g(s)}{f(s)} = \frac{b(s)}{a(s)}
\]

Exact model matching:

• Given the system \(a(s), b(s)\) and requested model, i.e. \(f(s), g(s)\)
• Find controller, i.e. \(p(s), q(s), r(s)\) so that
• resulting closed loop transfer function equals the model

Solution:

All such controllers fullfill

\[
a(s)p(s) + b(s)q(s) = f(s)\bar{b}(s)t(s) \\
r(s) = \bar{g}(s)t(s)
\]

• where \(\bar{b}(s)/\bar{g}(s) = b(s)/g(s)\) coprime
• \(t(s)\) is an arbitrary parameter