Sampled data systems. Discrete-time equivalents.

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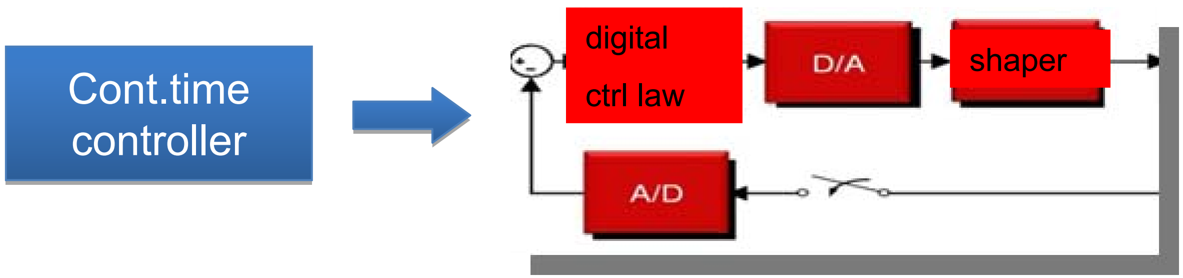
12-IV-12

Simple discretization procedures.



forward Euler rule (explicit)

$$\frac{dx(t)}{dt} \approx \frac{x(t+h)-x(t)}{h} \quad \Rightarrow \quad sx \approx \frac{z-1}{h}x \quad \Rightarrow \quad \boxed{s \approx \frac{z-1}{h}}$$

$$z = e^{sh} \approx 1 + sh$$

backward Euler rule (implicit)

$$\frac{dx(t)}{dt} \approx \frac{x(t)-x(t-h)}{h} \quad \Rightarrow \quad sx \approx \frac{z-1}{zh}x \quad \Rightarrow \quad \boxed{s \approx \frac{z-1}{zh}}$$

$$z = e^{sh} \approx \frac{1}{1-sh}$$

Tustin method

bilinear transformation

$$s \approx \frac{2}{h}\frac{z-1}{z+1} \qquad z = e^{sh} \approx \frac{1+sh/2}{1-sh/2}$$

(... compare to 1st order Pade approximation of time delay ...)

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>> c2d(f,h,'tustin')
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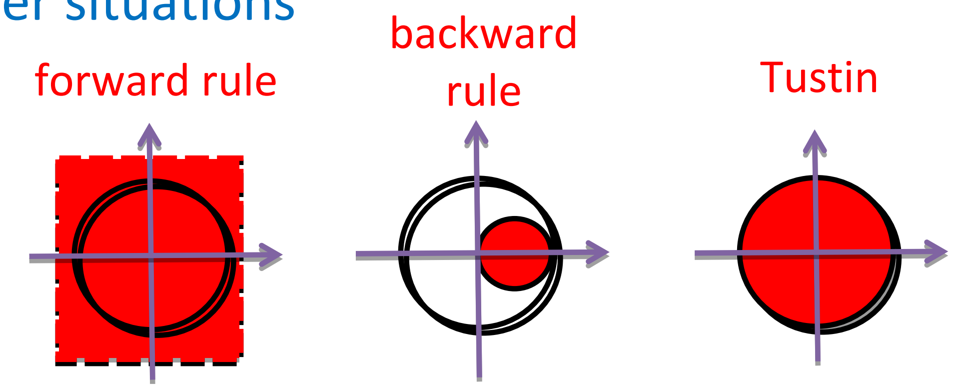
replace s with respective prescription

- suitable for simple pen-and-paper situations
- s-plane mappings:



- mind state space implications (A quite clear, B is affected as well though!)

Discrete-time PID controllers (PSD)

P:

$$u(t) = Ke(t) \Rightarrow u(s) = Ke(s) \Rightarrow u(z) = Ke(z) \Rightarrow u(k) = Ke(k)$$

I: forward rule

$$u(t) = \frac{K}{T_I}\int_0^t e(\tau)d\tau \Rightarrow u(s) = \frac{K}{T_I s}e(s) \Rightarrow u(z) = \frac{K}{T_I}\frac{h}{z-1}e(z) \Rightarrow$$

$$\Rightarrow u(k+1) = u(k) + \frac{Kh}{T_I}e(k)$$

D: backward rule

$$u(t) = KT_D\dot{e}(t) \Rightarrow u(s) = KT_D s e(s) \Rightarrow$$

$$u(z) = KT_D\frac{z-1}{zh}e(z) \Rightarrow u(k+1) = \frac{KT_D}{h}\left(e(k+1) - e(k)\right)$$

Discrete-time PID controllers (PSD)

- PSD

$$u(z) = K\left[1 + \frac{h}{T_I}\frac{1}{z-1} + \frac{T_D}{h}\frac{z-1}{z}\right]e(z)$$

- Alternatives:

$$u(s) = K\left[1 + \frac{1}{T_I s} + T_D s\right]e(s)$$

fwd.rule

$$u(z) = K\left[1 + \frac{h}{T_I}\frac{z}{z-1} + \frac{T_D}{h}\frac{z-1}{z}\right]e(z)$$

Tustin

$$u(z) = K\left[1 + \frac{h}{2T_I}\frac{z+1}{z-1} + \frac{2T_D}{h}\frac{z-1}{z+1}\right]e(z)$$

State feedback & discretization

$$\dot{x} = Ax + Bu,\ y = Cx$$
$$u(t) = Mu_C(t) - Kx(t)$$

$$K_{dis} = K\left(I + (A - BK)\,h/2\right)$$
$$M_{dis} = \left(I - KB\,h/2\right)M$$

-better correspondance of CL dynamics (in CT and DT) ...

-cont.time: A, B

A-BK

-discretization, Tustin rule:

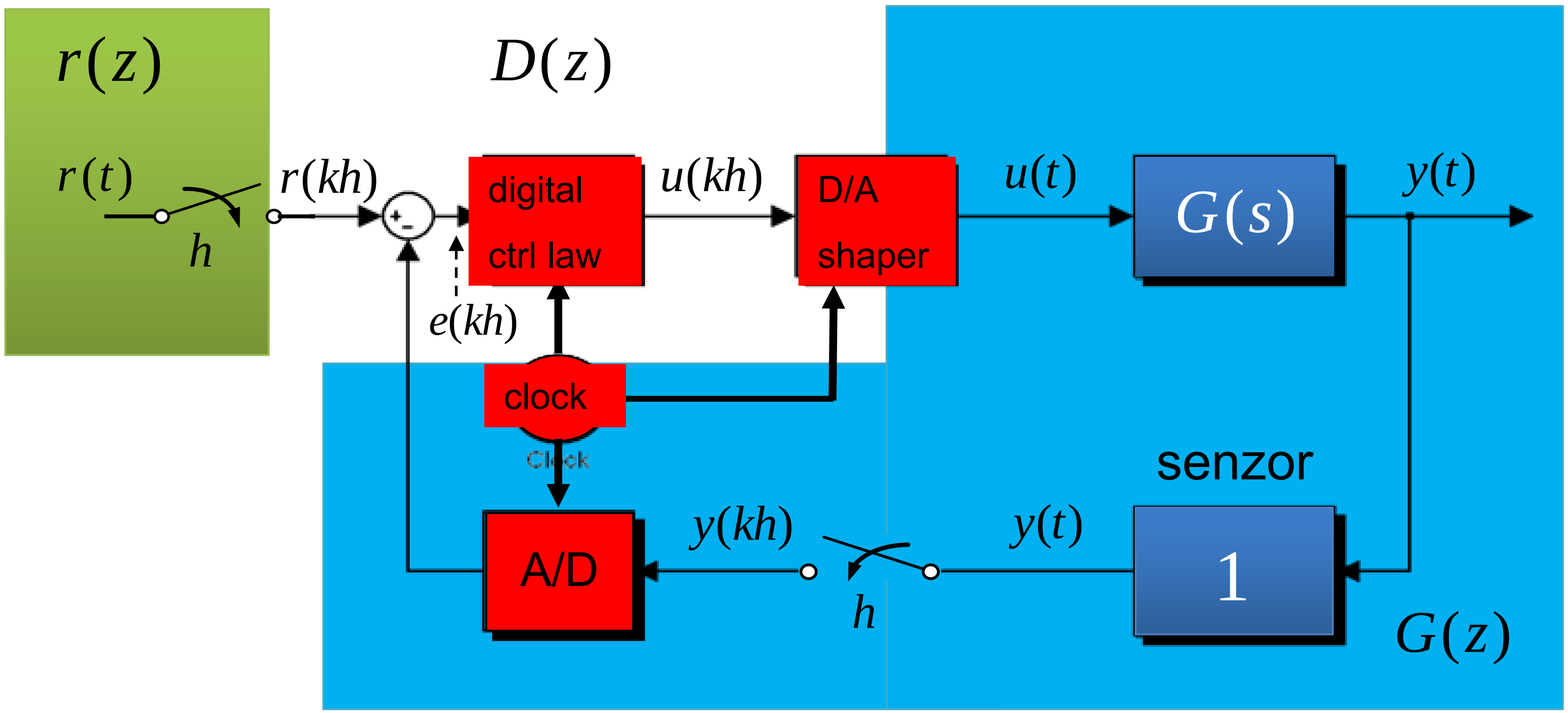
(I-Ah/2) \ (I+Ah/2), (I-Ah/2) \ B

(I-Ah/2) \ (I+Ah/2) - (I-Ah/2) \ B K(I+(A-BK)h/2) =

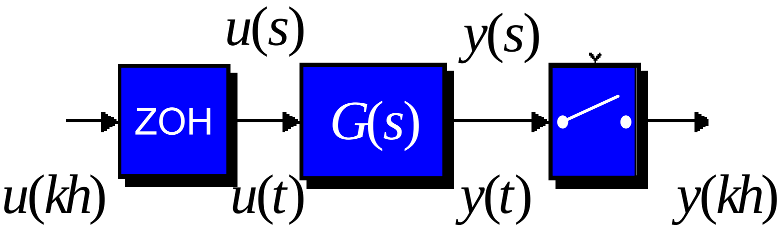
= (I-Ah/2) \ (I+Ah/2 – BK(I+(A-BK)h/2)

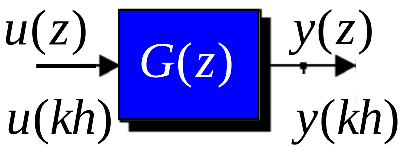
= (I-Ah/2) \ (I+Ah/2 – BK – BKAh/2 – BKBKh/2) ...

ZOH discretization.



ZOH





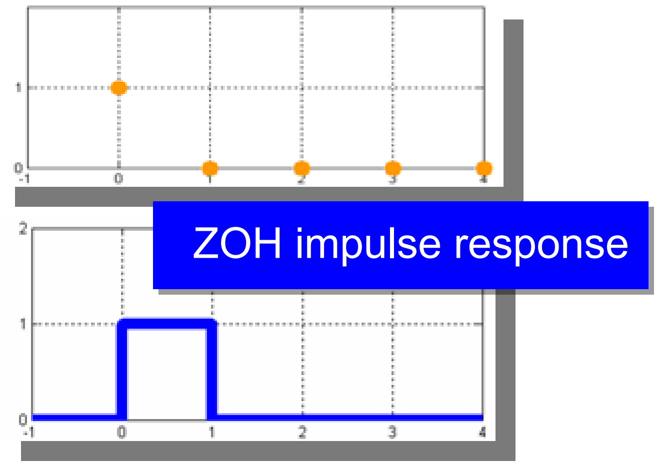
$$u(kh)=1 \quad k=0$$
$$u(kh)=0 \quad k\neq 0$$

ZOH

$$u(t)=1(t)-1(t-h) \qquad \frac{1}{s}-\frac{1}{s}e^{-hs}$$

system dynamics

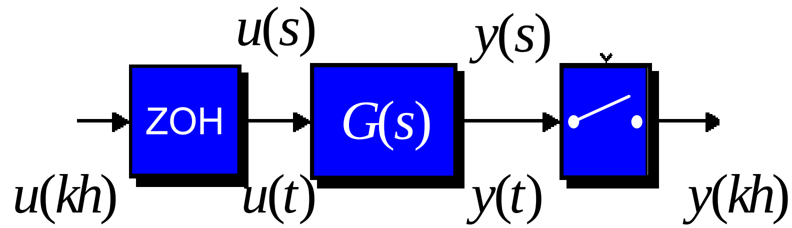
$$Y(s)=\left(1-e^{-hs}\right)\frac{G(s)}{s}$$



Discrete-time description

notation

$$G(z) = \mathcal{Z}\{y(kT)\} = \mathcal{Z}\{\mathcal{L}^{-1}\{Y(s)\}\} = \mathcal{Z}\{Y(s)\} = \mathcal{Z}\left\{(1-e^{-hs})\frac{G(s)}{s}\right\}$$



$$G(z) = \mathcal{Z}\left\{\frac{G(s)}{s}\right\} - \mathcal{Z}\left\{e^{-Ts}\frac{G(s)}{s}\right\}$$

$$\mathcal{Z}\left\{e^{-hs}\frac{G(s)}{s}\right\} \equiv z^{-1}\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$G(z) = (1-z^{-1})\,\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

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c2d(G,h,'zoh')
c2d(G,h)
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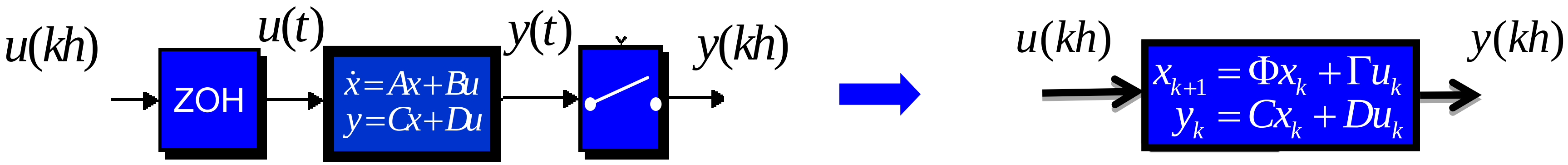
ZOH & state-space



$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

ZOH & state-space

$$t_k \longrightarrow t_{k+1}$$

$$u_k = u(\tau), \tau \in [t_k, t_{k+1})$$
$$h = t_{k+1} - t_k$$

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau$$
$$= e^{Ah}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}d\tau Bu(t_k)$$
$$= e^{Ah}x(t_k) + \left(\int_0^h e^{A\nu}d\nu\right)Bu(t_k) \qquad \nu = t_{k+1} - \tau$$

$$x(t_{k+1}) = \Phi x(t_k) + \Gamma u(t_k)$$
$$y(t_k) = Cx(t_k) + Du(t_k)$$

$$\Phi = e^{Ah}$$
$$\Gamma = \left(\int_0^h e^{A\nu}d\nu\right)B$$