Exercises for lectures
25 – Time-delay systems

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Time-delays in Laplace transform

\[ f(t) : f(t) = 0 \quad \forall t < 0, \quad \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt = f(s) \]

\[ f(t-\tau) : \tau > 0, \quad \mathcal{L}\{f(t-\tau)\} = ? \]

\[ \mathcal{L}\{f(t-\tau)\} = \int_{0^-}^{\infty} e^{-st} f(t-\tau) dt \leftarrow t - \tau = \nu \]

\[ = \int_{-\tau^-}^{\infty} e^{-s(\nu+\tau)} f(\nu) d\nu \]

\[ = e^{-s\tau} \int_{-\tau^-}^{\infty} e^{-sv} f(\nu) dt \]

\[ = e^{-s\tau} \int_{0^-}^{\infty} e^{-sv} f(v) dv \]

\[ = e^{-s\tau} f(s) \]

\[ \mathcal{L}\{\delta(t-\tau)\} = e^{-s\tau}, \mathcal{L}\{1(t-\tau)\} = \frac{e^{-s\tau}}{s}, \ldots \]
Example: Rolling mill

- time-delay at the output
- transfer function

\[ H(s, e^{-\tau s}) = G(s)e^{-\tau s} \]

- Therefore it contains dynamics without delay + delay
Phosphate extraction in Bou Craa, Western Sahara: 100 km conveyor system

Bangladesh: 1 belt 17 km

Example: Belt conveyor
Example: Regulation in a cell

\[
\begin{align*}
\dot{x}_1(t) &= -\lambda_1 x_1(t) + c_1 x_2(t - \tau_1) \\
\dot{x}_2(t) &= -\lambda_2 x_2(t) + g(x_1(t - \tau_1))
\end{align*}
\]

**REGULATORY NETWORKS**

Cyclic biochemical feedback in cell regulatory networks is affected by delays. Consider the model

\[
\begin{align*}
\dot{x}_1(t) &= -\lambda_1 x_1(t) + c_1 x_2(t - \tau_1), \\
\dot{x}_2(t) &= -\lambda_2 x_2(t) + g(x_1(t - \tau_2))
\end{align*}
\]

(S5)  
(S6)

where \(x_1\) denotes the concentration of the messenger RNA (mRNA), \(x_2\) denotes the concentration of the protein, which is the end product of the reaction, and the rate \(\dot{x}(t)\) is defined by the balance between mRNA synthesis and the end product consumption [S14]. The delays \(\tau_1\) and \(\tau_2\), respectively, define the lag from the initiation of the translation and from the initiation of the transcription until the appearance of the mature protein mRNA, \(c_1 > 0\) describes the translation effects, \(\lambda_1 > 0\) and \(\lambda_2 > 0\) are related to degradation effects, and \(g\) is the feedback function.

System (S5)–(S6) is an example of a low-order biochemical oscillator model, where delays describe chemical or biochemical...
EPIDEMICS

Understanding the underlying mechanisms of biological processes and epidemics represents a challenge for health workers engaged in designing clinically relevant treatment strategies. These mechanisms can be revealed by considering epidemics and diseases as dynamical processes.

Hematology dynamics can be modeled by

\[ \dot{x}(t) = -\lambda x(t) + G(x(t-\tau)), \] (S7)

which formulates the circulating cell populations in one compartment, where \( x \) represents the circulating cell population, \( \lambda \) is the cell-loss rate, and the monotone function \( G \), which describes a feedback mechanism, denotes the flux of cells from the previous compartment [61]. The delay \( \tau \) represents the average length of time required to go through the compartment. Model (S7) is also found in population dynamics, where the delay represents a maturation period.

Models representing regulatory feedback mechanisms in the production of blood cells are similar to (S7). An example is the characteristic equation of the linearized system

\[ f(s; \tau, \lambda, \lambda_E, k) = (s + \lambda)(s + \lambda_E)(s + k)e^{-s\tau} = 0, \] (S8)

where \( \lambda > 0 \) is the death rate, \( \lambda_E > 0 \) is the decay constant of a hormone at the equilibrium of the dynamics, and \( \tau \) is the length of time needed for the maturation of red-blood-cell precursors [S23].
Example: Enzyme activation mechanism

**FIGURE S4** Block diagram of enzyme-activation mechanisms. A cascade of systems is used in [S20] to model the enzyme-activation mechanisms with delays. In this model, the production rate of the enzyme $E_j$ depends on the production rate of the enzyme $E_{i-1}$. The effect of $E_{i-1}$, however, takes place after a length of time $\tau_{i-1}$ elapses. In a biological system, the variable $x_i$ may represent the amount of enzyme $E_j$ available at time $t$, while $G_j$ and $H_{\tau_i}$ represent, respectively, nonlinear dynamics with outputs $x_i$ and $y_i$. Moreover, the action $u$ on $G_1$ can be inhibited by the final product $x_n$. The closed-loop system may oscillate or exhibit chaos.
Examples: Operational research

**FIGURE S5** Supply chains and delays. Supply-chain systems are examples of interconnected supply-demand points, which share products and information to regulate inventories and optimally respond to customer demands. Various sources of delay in supply chains include decision-making delays, transportation lines, and lead times in manufacturing facilities. Delays in supply chains influence every stage of the supply-demand chain, causing financial losses, inefficiencies, and reduced quality of service.

**FIGURE S6** Inventory acquisition model [S27]. This model represents the flow of products in a supply chain, where decision-making adjusts the orders needed to respond to each customer’s buying rate, that is, the loss rate. Due to the presence of delays, the orders placed earlier by the decision maker traverse the supply line first and then arrive at the inventory after a delay.
Example: Thermal system

\[
T_h \dot{x}_h(t) = -x_h(t - \eta_h) + K_b x_a(t - \tau_b) + K_{u} x_{h,\text{set}}(t - \tau_{u})
\]

\[
T_a \dot{x}_a(t) = -x_a(t) + x_c(t - \tau_e) + K_a \left( x_h(t) - \frac{1+q}{2} x_a(t) - \frac{1-q}{2} x_c(t - \tau_e) \right)
\]

\[
T_d \dot{x}_d(t) = -x_d(t) + K_d x_a(t - \tau_d)
\]

\[
T_c \dot{x}_c(t) = -x_c(t - \eta_c) + K_c x_d(t - \tau_c)
\]

\[
\dot{x}_e(t) = x_{c,\text{set}}(t) - x_c(t)
\]

Linear system of dimension 6, 5 delays.,
Example: Network control systems

FIGURE 2 Network control systems. Controlling across a shared communication network is a challenging task due to the delays arising in the communication medium. Delays can manifest themselves in the control signals, in the measured signals, and in external inputs traveling from their source to their destination through the links of the network.
I. Fluid flow model for a congested router in TCP/AQM controlled network

Model of collision-avoidance type: Hollot et al., IEEE TAC 2002

\[ \dot{W}(t) = \frac{1}{R(t)} - \frac{1}{2} \frac{W(t)W(t-R(t))}{R(t-R(t))} p(t-R(t)) \]

\[ \dot{Q}(t) = \begin{cases} N(t) \frac{W(t)}{R(t)} - C & Q > 0 \\ \max \left( N(t) \frac{W(t)}{R(t)} - C, 0 \right) & Q = 0 \end{cases} \]

\[ R(t) = \frac{Q(t)}{C} + T_p \]

Packet marking

Normalization of state and time

\[ \dot{W}(t) = \frac{1}{R} - \frac{1}{2} \frac{W(t)W(t-R)}{R} K Q(t-R) \]

\[ \dot{Q}(t) = \begin{cases} N \frac{W(t)}{R} - C & Q > 0 \\ \max \left( N \frac{W(t)}{R} - C, 0 \right) & Q = 0 \end{cases} \]

\[ w = W, \quad q = \frac{Q}{N}, \quad t^{(new)} = t^{(old)} + \frac{1}{R} \]

\[ \dot{w}(t) = 1 - \frac{1}{2} w(t) w(t-1) k q(t-1) \]

\[ \dot{q}(t) = \begin{cases} w(t) - c & q > 0 \\ \max (w(t) - c, 0) & q = 0 \end{cases} \]

Linearized model

\[ \dot{w}(t) = 1 - \frac{1}{2} w(t) w(t-1) k q(t-1) \]

\[ \dot{q}(t) = \begin{cases} w(t) - c & q > 0 \\ \max (w(t) - c, 0) & q = 0 \end{cases} \]

Unique steady state solution \((w^*, q^*) = (c, \frac{2}{k c^2})\)

Linearization:

\[ \ddot{q}(t) + \frac{1}{c} \dot{q}(t) + \frac{1}{2} \dot{q}^2(t-1) + \frac{k c^2}{2} q(t-1) = 0 \]

\[ \lambda^2(t) + \frac{1}{c} \lambda(t) + \frac{1}{2} \lambda e^{-\lambda} + \frac{k c^2}{2} e^{-\lambda} = 0 \]
III. Rotating cutting and milling machines

Milling process

\[
\begin{align*}
\dot{x}(t) &= A(\omega t) x(t) + B(\omega t) x(t - \tau(t)) \\
\tau(t) &= \tau_0 + \delta f(\Omega t)
\end{align*}
\]

- Successive passage of teeth \(\Rightarrow\) delay
- Rotation of each tooth \(\Rightarrow\) periodic coefficients

Cutting process

- Successive passage of the same point of the piece \(\Rightarrow\) delay
- Orientation of tooth w.r.t. workpiece is fixed \(\Rightarrow\) constant coefficients

Both cases: speed determines delay

Unstable steady state goes to chatter or oscillations of workpiece/tool \(\Rightarrow\) irregular surface
\[ D(s) = e^{-s} \]
Input / Output Delayed System in Matlab

\[ G(s) = \frac{1}{10 + s} e^{-2.1s} \]

\[
\begin{align*}
\text{Transfer function:} \\
&= \frac{1}{s + 10} \\
&= \frac{\exp(-2.1s)}{s + 10}
\end{align*}
\]
System with internal delay in Matlab

\[ T(s) = \frac{1}{10 + s} e^{-2.1s} \]

\[ T(s) = \frac{1}{10 + s} + \frac{e^{-2.1s}}{10 + s} e^{-2.1s} \]

\[ = \frac{1}{10 + s + e^{-2.1s}} \]

\[ \text{Output delays (seconds): 2.1} \]
\[ \text{Internal delays (seconds): 2.1} \]
\[ \text{Continuous-time model.} \]

\[
\begin{align*}
>> s &= \text{tf('s'); GG = 1/(10+s)}; \\
>> G &= \text{exp(-2.1*s)*GG; } \\
>> T &= G/(1+G) \\
... \\
\end{align*}
\]

\[
\begin{align*}
\text{>> step(T)}
\end{align*}
\]
Example: More complex systems

\[ H(s, e^{-s}) = \frac{e^{-s}}{s + 1 + se^{-s}} \]

```matlab
>> delay=tf(1);set(delay,'ioDelay',1)
>> E=tf(delay),S=tf(s);
Transfer function:
exp(-1*s) * (1)
>> H=E/(S+a+S*b*E),step(H,20)
```
Example: More complex systems

\[ H(s, e^{-0.2s}, e^{-0.3s}) = \frac{e^{-0.2s}}{s + e^{-0.3s} + 2e^{-0.2s}} \]

\[ \begin{align*}
\text{Transfer function:} \\
&\exp(-0.2s) \cdot (1) \\
\text{Transfer function:} \\
&\exp(-0.3s) \cdot (1) \\
\end{align*} \]

\[ H = \frac{E2}{S+E3+2E2}; \text{step}(H, 20) \]

\[ E2 = \text{tf}(\text{delay2}), E3 = \text{delay3}, S = \text{tf}(s); \]

\[ \text{Nyquist Diagram, Bode Diagram, Step Response Graph} \]
Interestingly: Lambert's function

Lambert's W-funkce (also omega function)

- Is inverse to \( f(W) = We^W \)
- In real field reasonable, but in complex wild (endlessly many branches)

\[
W(f) = We^W
\]

real and imaginary part of Lambert's function (analytical extension)

Example: Retarded and neutral system

• Retarded system

\[ c_{CL}(s) = s + e^{-s} \]

• Neutral system

\[ c_{CL}(s) = e^{-s} s + 1 \]
Conditions of stability of a simple quasipolynomial

(Kharitonov et al., 2004, p. 40)

- System with characteristic quasipolynomial

\[ a(s, e^{-\tau s}) = s + a + be^{-\tau s} \]

where \( a + b > 0 \) (If not, then it is not stable without delay)

- Stable regardless of the delay size („iod“)

iff \( a \geq |b| \). Else

- If \( a > 0, b > 0 \), it is stable

\[ \forall \tau < \frac{\pi - \arccos \left( \frac{a}{b} \right)}{\sqrt{b^2 - a^2}} \]

- If \( a < 0, b > 0 \), it is stable

\[ \forall \tau < \frac{\arccos \left( \frac{|a|}{b} \right)}{\sqrt{b^2 - a^2}} \]
Example: Destabilizing effect of delay

\[ G(s) = \frac{1}{s}, \quad C(s) = Ke^{-\tau s} \]

\[ c_{CL}(s) = s + Ke^{-\tau s} \]

\[ \tau = 0: \quad s_i = -K \]
\[ \tau = 0^+: \quad \max \{\text{Re} s_i\} \ll 0 \]
\[ \tau = \uparrow: \quad \max \{\text{Re} s_i\} \uparrow 0 \]

\[ \tau = 0, \quad \tau = 0^{++}, \quad \tau = 0^{+} \]

\[
\begin{align*}
\text{max } \text{Re} \{s_i\} & \quad \tau_c = 1.5708 \\
\tau = 0.5 & \\
\tau = 1 & 
\end{align*}
\]

\[ >> \text{solve('x+k*exp(-tau*x)=0')} \]
\[ \text{ans} = \text{lambertw}(0, -k*tau)/tau \]
\[ >> k=1, \tau=0.5 \]
\[ >> r1=\text{lambertw}(-10:10, -k*tau)/tau; \]
\[ >> \text{plot}(\text{real}(r), \text{imag}(r), '+r') \]

\[ \tau_c \]

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Example: Stabilizing effect of delay

\[ c_{CL}(s) = s^2 + 9 + 1.5e^{-\tau s} \]

- is unstable for \( \tau = 0 \),
- but stable for small non-zero delays

• Compare its response using the PD controller

**FIGURE 11** Step response. The positive feedback control loop consists of the open-loop transfer function \( H(s) = 1/(s^2 + 9) \) and the controller \( C(s) = (k_p s + k_d s) \). The aim is to compare the speed of response between a delay-free proportional-derivative controller (\( k_p \neq 0, k_d \neq 0, \tau = 0 \)) and a delayed proportional controller (\( k = k_p \neq 0, \tau = 0, \) and \( k_d = 0 \)). Curve 1 denotes the case where there is no delay in the closed-loop system with the controller gains \( k_p = 7 \) and \( k_d = -2 \). Curve 2 corresponds to the output of the system with \( \tau = 0.3 \) s and the proportional controller gain \( k = 7 \). Curve 3 represents the output of the system with no delay and controller gains \( k_p = 7 \) and \( k_d = -3 \). Finally, curve 4 denotes the output of the system with delay \( \tau = 0.6 \) s and controller gain \( k = 7 \).
Example: Delay as a derivative FB

- System with equation \( \dot{y}(t) - 0.1\dot{y}(t) + y(t) = u(t) \) is unstable.
- We can stabilize it with derivative FB and gain \( k > 0.1 \)
  \[ u(t) = k\dot{y}(t) \]
- Alternatively, we can stabilize it „delayed ZV“
  \[ u(t) = y(t - \tau) - y(t) \]
- Which we can interpret as BF with the final difference
  \[ u(t) = -\tau \left( \frac{y(t) - y(t - \tau)}{\tau} \right) \]
- Approximate derivation with \( k = \tau \)
Henri Eugene Padé - French mathematician (1863-1953)

Today known mainly as the author of the approximation of the general function by means of a rational function,

which is often better than Taylor.

Pade Approximation

For given function $f$ and natural numbers $m,n$ is Pade approximation of order $(m,n)$

$$R(x) = \frac{p_0 + p_1 x + p_2 x^2 + \cdots + p_m x^m}{1 + q_1 x + q_2 x^2 + \cdots + q_n x^n}$$

where

$$f(0) = R(0)$$

$$f'(0) = R'(0)$$

$$f''(0) = R''(0)$$

$$\vdots$$

$$f^{(m+n)}(0) = R^{(m+n)}(0)$$

The sum of the first $m+n+1$ elements of Taylor series $f$ and $R$ is the same.
Example: Pade Approximation

\[
\begin{align*}
>> \text{del} & = \text{tf}(1); \\
>> \text{set}([\text{del}, 'ioDelay'],5); & \\
\quad \text{Transfer function:} \\
\quad \exp(-5s) \ast 1 \\
>> \text{pade1} & = \text{pade}([\text{del},1]) \\
\quad \text{Transfer function:} \\
\quad -s + 0.4 \\
\quad ----------- \\
\quad s + 0.4 \\
>> \text{pade2} & = \text{pade}([\text{del},2]) \\
\quad \text{Transfer function:} \\
\quad s^2 - 1.2s + 0.48 \\
\quad ------------------ \\
\quad s^2 + 1.2s + 0.48 \\
>> \text{pade3} & = \text{pade}([\text{del},3]) \\
\quad \text{Transfer function:} \\
\quad -s^3 + 2.4s^2 - 2.4s + 0.96 \\
\quad --------------------------- \\
\quad s^3 + 2.4s^2 + 2.4s + 0.96 \\
>> \text{step}([\text{del}, \text{pade1}, \text{pade2}, \text{pade3}]) \\
>> \text{bode}([\text{del}, \text{pade1}, \text{pade2}, \text{pade3}])
\end{align*}
\]
Example for "exact" design: P regulator

- for \[ G(s) = \frac{K}{Ts+1}e^{-\tau s}, C(s) = K_p \] \[ T(s) = \frac{KK_p e^{-\tau s}}{(Ts + 1) + KK_p e^{-\tau s}} \]
- and for values \( K = T = \tau = 1, K_p = 2 \)
- CL characteristic quasipolynomial is
- For \( c_{CL}(s) = s + 1 + 2e^{-s} \) \[ K_p = 5 \quad c_{CL}(s) = s + 1 + 5e^{-s} \]

```
>> solve('x+1+5*exp(-x)=0')
ans = -1+lambertw(-5*exp(1))
>> r=-1+lambertw(-10:10,-5*exp(1));
>> plot(real(r),imag(r),'*')
>> solve('x+1+2*exp(-x)=0')
ans = -1+lambertw(-2*exp(1))
>> r=-1+lambertw(-10:10,-2*exp(1));
>> plot(real(r),imag(r),'*')
```
Example: CL stability

\[ G(s) = \frac{b}{s + a} e^{-\tau s} \]

\[ C(s) = k \]

\[ T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)} \]

\[ = \frac{kb}{s + a} e^{-\tau s} \]

\[ = \frac{kb e^{-\tau s}}{s + a + kbe^{-\tau s}} \]

![Stability chart](image)

**FIGURE 4** Stability chart. This chart is obtained for a closed-loop system with the plant transfer function \( e^{-\tau s}b/(s + a) \) and the controller \( C(s) = k \). This stability chart is partitioned into three regions, namely, delay-independent stable, delay-dependent stable, and unstable. This chart reveals the effect of a delay parameter on stability and how the controller gain \( k \) can be tuned to avoid instability.
Example: Precise design with an I controller

- For
  \[ G(s) = \frac{K}{Ts + 1}, \quad C(s) = \frac{1}{T_I s} \]
- is
  \[ T(s) = \frac{Ke^{-\tau s}}{T_I s(Ts + 1) + Ke^{-\tau s}} \]
- and for values \( K = T_I = T = \tau = 1 \)
  \[ T(s) = \frac{e^{-s}}{s(s+1) + e^{-s}} \]
- CL characteristic quasipolynomial is
  \[ c_{CL}(s) = s^2 + s + e^{-s} \]
- Has infinitely many roots
  Part of the roots above the real axis

\[ \begin{array}{c}
1.0e+002 * \\
-0.09225032465054+1.00357526290690i \\
-0.09096327458520+0.94065517770892i \\
-0.08958771536361+0.87772450636321i \\
-0.08811056322665+0.81478112589439i \\
-0.08651559672926+0.75182288048141i \\
-0.08478236381840+0.68884436592290i \\
-0.08288456611943+0.62584246447846i \\
-0.08078758735029+0.56280979999999i \\
-0.07844455447831+0.499736666852271i \\
-0.07578973834359+0.43660864096520i \\
-0.07272678028205+0.37340319845989i \\
-0.06910590713751+0.31008293641609i \\
-0.06467468145853+0.24658031170133i \\
-0.05895295773482+0.18275807028892i \\
-0.05081944749196+0.11828269472765i \\
-0.03610143271894+0.05213342976426i \\
-0.00037250765679+0.00819937714011i
\end{array} \]
Example: Smith's predictor

- P controller with gain 2

\[
T(s) = \frac{2}{s + 1 + 2e^{-s}e^{-s}}
\]
Example: Smith's predictor

- **P controller with gain 5**

\[ T(s) = \frac{5}{s + 6}e^{-s} \]

- **After disconnecting the predictor**

\[ T(s) = \frac{5}{s + 1 + 5e^{-s}} \]
Example: Smith's predictor

- I controller and Smith's predictor
• Smith's predictor does not work for an unstable system! Why?
• But for example, 
  \[ G(s) = \frac{b(s)}{a(s)} = \frac{e^{-0.5s}}{s-1} \] can still be stabilized.

• For this particular delay, "iod" it does not work.
• P-controller with gain \( k = 1.5 \) gives stable system.

\[
c_{CL}(s) = s - 1 + 1.5e^{-0.5s}
\]

\[
T(s) = \frac{1.5e^{-0.5s}}{s-1 + 1.5e^{-0.5s}}
\]
Example: Assign the final number of poles

- **Unstable system**

\[
G(s) = \frac{b(s)}{a(s)} = \frac{1 + e^{-s}}{s - e^{-s}}
\]

- **Char. kvazipolynom**

\[
(s - e^{-s})p(s) + (1 + e^{-s})q(s) = s + 1
\]

- **Controller and CL**

\[
\frac{q(s)}{p(s)} = \frac{1}{1} \rightarrow T(s) = \frac{1 + e^{-s}}{s + 1}
\]

- **For simulation \texttt{pade}(3, 3)**

\[
e^{-s} = \frac{120 - 60s + 12s^2 - s^3}{120 + 60s + 12s^2 + s^3}
\]