5 – Systems interconnections.

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Automatic control 2012
\[ y_i(s) = F_i(s)u_i(s) \quad F_i(s) = \frac{b_i(s)}{a_i(s)} \]

\[ \dot{x}_i = A_i x_i + B_i u_i \quad y_i = C_i x_i + D_i u_i \]

\[ a_i(s) = \det(sI - A_i) \quad i = 1, 2 \]

\[ y(s) = F(s)u(s) \quad F(s) = \frac{b(s)}{a(s)} \]

\[ \dot{x} = Ax + Bu \]

\[ y = Cx + Du \]

\[ a(s) = \det(sI - A) \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]
Serial connection (cascade)

\[ y(s) = y_2(s) = F_2(s)u_2(s) = F_2(s)y_1(s) = F_2(s)F_1(s)u_1(s) = F_2(s)F_1(s)u(s) \]
\[ y(s) = F(s)u(s), F(s) = F_2(s)F_1(s) \]

\[ \frac{b(s)}{a(s)} = \frac{b_2(s)b_1(s)}{a_2(s)a_1(s)} \]

\[ \dot{x}_1 = A_1x_1 + B_1u_1 = A_1x_1 + B_1u \]
\[ \dot{x}_2 = A_2x_2 + B_2u_2 = A_2x_2 + B_2 \left( C_1x_1 + D_1u \right) \]
\[ y = y_2 = C_2x_2 + D_2u_2 = C_2x_2 + D_2C_1x_1 + D_2D_1u \]

\[ a(s) = a_1(s)a_2(s) = \det(sI - A) = \det(sI - A_1) \det(sI - A_2) \]
Paralel connection

\[ y(s) = y_1(s) + y_2(s) = (F_1(s) + F_2(s))u(s) \]
\[ y(s) = F(s)u(s), F(s) = F_1(s) + F_2(s) \]
\[ \frac{b(s)}{a(s)} = \frac{b_1(s)}{a_1(s)} + \frac{b_2(s)}{a_2(s)} = \frac{a_2(s)b_1(s) + a_1(s)b_2(s)}{a_1(s)a_2(s)} \]

\[
\begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
x \\
u
\end{bmatrix}
= 
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}u
\]
\[
y = [C_1 & C_2] \begin{bmatrix} x \\ (D_2 + D_1)u
\end{bmatrix}
\]
\[ a(s) = a_1(s)a_2(s) = \det(sI - A) = \det(sI - A_1)\det(sI - A_2) \]
Feedback(-s)

\[ y(s) = F_1(s)(u(s) - y_2(s)) = F_1(s)(u(s) - F_2(s)u_2(s)) = F_1(s)u(s) - F_1(s)F_2(s)y(s) \]

\[ (1 + F_1(s)F_2(s))y(s) = F_1(s)u(s) \]

\[ y(s) = \frac{F_1(s)}{1 + F_1(s)F_2(s)}u(s) \]

\[ F_2(s) = 1 \]

\[ F(s) = \frac{F_1(s)}{1 + F_1(s)} \]

\[ \frac{b(s)}{a(s)} = \frac{b_1(s)}{a_1(s) + b_1(s)} \]

\[ f(s) = \frac{F_1(s)}{1 + F_1(s)F_2(s)} \]

\[ \frac{b(s)}{a(s)} = \frac{a_2(s)b_1(s)}{a_1(s)a_2(s) + b_1(s)b_2(s)} \]
\[ D_1 = 0, \quad D_2 = 0 \]

\[ \dot{x} = \begin{bmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \]

\[ y = \begin{bmatrix} C_1 & 0 \end{bmatrix} x \]

\[ u_1 = u - y_2 \]

\[ y_1 = u_2 = y \]

\[ c(s) = \det(sI - A) = \det(sI - A_1) \det(sI - A_2) \det \left( I - C_2 (sI - A_2)^{-1} B_2 C_1 (sI - A_1)^{-1} B_1 \right) \]

\[ F_2(s) \]

\[ F_1(s) \]
2 DoF controller

\[ y = G(u + d) + n \]
\[ u = K(Fr - y) \]

\[ y = \frac{GKF}{1+GK}r + \frac{G}{1+GK}d + \frac{1}{1+GK}n \]
\[ \eta = \frac{GKF}{1+GK}r + \frac{G}{1+GK}d - \frac{GK}{1+GK}n \]
\[ \nu = \frac{KF}{1+GK}r + \frac{1}{1+GK}d - \frac{K}{1+GK}n \]
\[ u = \frac{KF}{1+GK}r - \frac{GK}{1+GK}d - \frac{K}{1+GK}n \]
\[ e = \frac{F}{1+GK}r - \frac{G}{1+GK}d - \frac{1}{1+GK}n \]

\[ \varepsilon = r - \eta = \left(1 - \frac{GKF}{1+GK}\right)r - \frac{G}{1+GK}d + \frac{GK}{1+GK}n \]

„Gang of Six“

\[ TF = \frac{GKF}{1+GK}, \quad T = \frac{GK}{1+GK}, \quad GS = \frac{G}{1+GK} \]
\[ KFS = \frac{KF}{1+GK}, \quad KS = \frac{K}{1+GK}, \quad S = \frac{1}{1+GK} \]
1 DoF controller

\[ y = G(u + d) + n \]
\[ u = K(r - y) \]

\[ y = \frac{GK}{1+GK}r + \frac{G}{1+GK}d + \frac{1}{1+GK}n \]
\[ \eta = \frac{GK}{1+GK}r + \frac{G}{1+GK}d - \frac{GK}{1+GK}n \]
\[ \nu = \frac{K}{1+GK}r + \frac{1}{1+GK}d - \frac{K}{1+GK}n \]
\[ u = \frac{K}{1+GK}r - \frac{GK}{1+GK}d - \frac{K}{1+GK}n \]
\[ e = \frac{1}{1+GK}r - \frac{G}{1+GK}d - \frac{1}{1+GK}n \]

\[ \varepsilon = r - \eta = \left(1 - \frac{GK}{1+GK}\right)r - \frac{G}{1+GK}d + \frac{GK}{1+GK}n \]

„Gang of Four“

\[ T = \frac{GK}{1+GK}, \quad GS = \frac{G}{1+GK} \]
\[ KS = \frac{K}{1+GK}, \quad S = \frac{1}{1+GK} \]
Struktura s jedním regulátorem

- Open loop
  \[ L = GK \]

- Sensitivity (response to dist.)
  \[ S = \frac{1}{1 + GK} \]

- Complementary sensitivity (CL transfer function)
  \[ T = \frac{GK}{1 + GK} \]

- Input sensitivity
  \[ GS = \frac{G}{1 + GK} \]

- Output sensitivity
  \[ KS = \frac{K}{1 + GK} \]

\[ y = Tr + GSd + Sn \]
\[ \eta = Tr + GSd - Tn \]
\[ u = KSr - Td - KSn \]
\[ e = Sr - GSd - Sn \]
\[ \varepsilon = Sr - GSd + Tn \]
Closed loop dynamics

\[ G(s) = \frac{b(s)}{a(s)}, \quad K(s) = \frac{q(s)}{p(s)} \]

\[ L(s) = G(s)K(s) = \frac{b(s)q(s)}{a(s)p(s)} \]

\[ S(s) = \frac{1}{1 + L(s)} = \frac{a(s)p(s)}{a(s)p(s) + b(s)q(s)} \]

\[ T(s) = \frac{L(s)}{1 + L(s)} = \frac{b(s)q(s)}{a(s)p(s) + b(s)q(s)} \]

\[ c(s) = a(s)p(s) + b(s)q(s) \]

Observations:

- unstable poles resp. zeros of \( L \) always appear as unstable zeros of \( S \) and \( T \) respectively (can never be cancelled). Leading to non-minimum phase CL system.
- unstable poles cannot be eliminated (while unstable poles can be stabilized by FB)
Exercise (HW): further interconnections

- Devise the following formulas:

\[ T_{ry} = \frac{G_2(KF_m + F_u)}{1 + GK} \]

\[ T_{ry} = \frac{G(1 + F_d G_1)}{1 + GK} = F_m + \frac{G F_u - F_m}{1 + GK}, \quad G = G_1 G_2 \]

\[ F_u \approx \frac{F_m}{G}, \quad F_d \approx -\frac{1}{G} \]

- Similarly for: