Stability of hybrid systems

Common Lyapunov function approach

Zdeněk Hurák

November 2023

Short recap of stability analysis for continuous dynamical systems

- Consider the standard state equation $\dot{x} = f(x)$.

Equilibrium

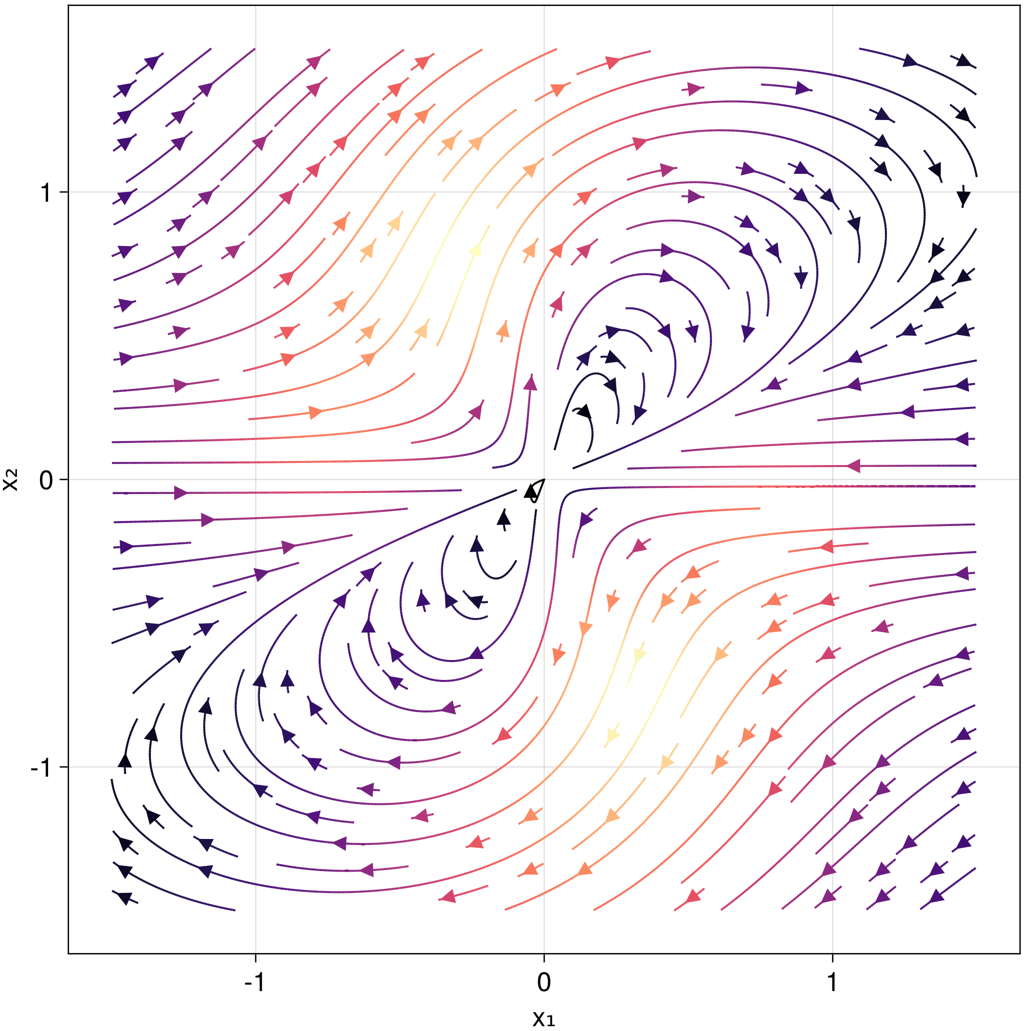
- x_{eq} such that $\dot{x} = 0 = f(x_{\mathrm{eq}})$.
  - Without loss of generality $x_{\mathrm{eq}} = 0$.

(Lyapunov) stability

- Starts close, stays close.
- For a given $\varepsilon > 0$, there is a $\delta > 0$ such that…

Attractivity

- Starts close, asymptotically converges.
- Global attractivity: asymptotically converges from anywhere.
- Example of attractive but unstable



Asymptotic stability

- Stability and attractivity.
- Global assymptotic stability: attractivity is global.

Exponential stability

- Exponential convergence.

Stability of time-varying systems

- Uniform (Lyapunov, Asymptotic, …): If independent of the inititial time.

Stability analysis via Lyapunov function

$V(\cdot) \in \mathcal{C}_1$ defined on open $\mathcal{D} \subset \mathbb{R}^n$ containing the origin.

$$V(0) = 0,\ V(x) > 0 \text{ for all } x \in \mathcal{D} \setminus \{0\}$$

$$\underbrace{\left(\nabla V(x)\right)^\top f(x)}_{\frac{\mathrm{d}}{\mathrm{d}t}V(x)} \leq 0$$

or

$$\left(\nabla V(x)\right)^\top f(x) < 0$$

Formulated using comparison functions

$$\kappa_1(\|x\|) \leq V(x) \leq \kappa_2(\|x\|),$$

- where $\kappa_1(\cdot), \kappa_2(\cdot)$ are class $\mathcal{K}$ comparison functions:
  - continuous, zero at zero and (strictly) increasing.
- If κ_1 increases to infinity ($\kappa_1(\cdot) \in \mathcal{K}_\infty$), the stability is global.

For asymptotic stability

$$\left(\nabla V(x)\right)^\top f(x) \leq -\rho(\|x\|),$$

where $\rho(\cdot)$ is a positive definite continuous function, zero at the origin.

Exponential stability

$$k_1\|x\|^p \leq V(x) \leq k_2\|x\|^p,$$

$$\left(\nabla V(x)\right)^\top f(x) \leq -k_3\|x\|^p.$$

Exponential stability with quadratic Lyapunov function

$$V(x) = x^\top P x$$

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2$$

Converse theorems

- for (G)UAS,
- for Lyapunov stability only time-varying Lyapunov function guaranteed.

Equilibrium of a hybrid automaton

- For a hybrid system with $\dot{x} = \underbrace{f(q,x)}_{f_q(x)}$, the equilibrium x_{eq} satisfies
  - $0 = f_q(x_{\mathrm{eq}})$ for all $q \in \mathcal{Q}$,
  - the reset map $r(q, q', x_{\mathrm{eq}}) = x_{\mathrm{eq}}$
- With no loss of generality $x_{\mathrm{eq}} = 0$.

Stability of a hybrid automaton

- The equilibrium x_{eq}(=0) is stable if for a given $\varepsilon > 0$ there exists $\delta > 0$ such that for all hybrid systems executions/trajectories starting at (q_0, x_0),

$$\|x_0\| < \delta \Rightarrow \|x(\tau)\| < \varepsilon,\ \forall \tau \in \mathcal{T},$$

  - where τ is a hybrid time and $\mathcal{T}$ is the hybrid time domain.

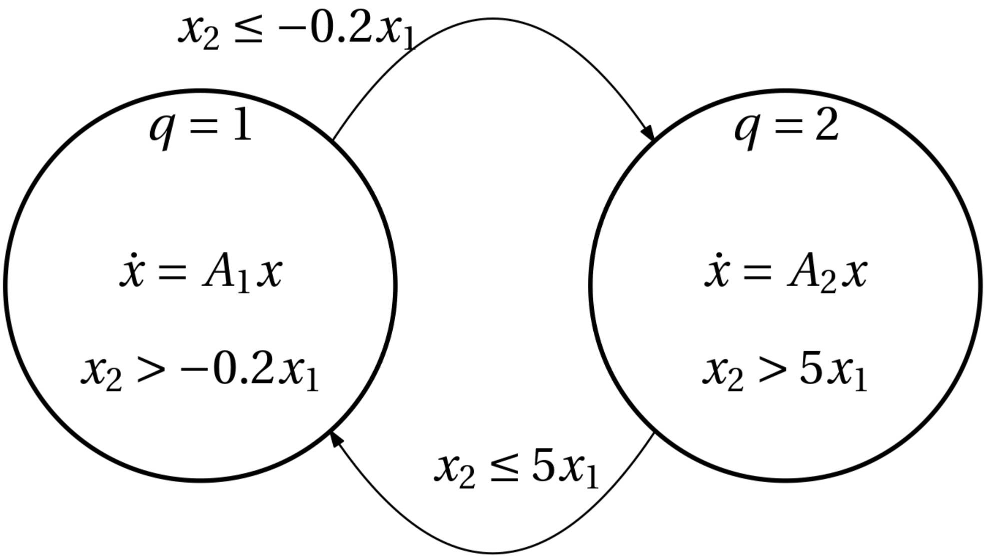
Asymptotic stability

- The equilibrium is stable and furthermore we can choose some δ such that

$$\|x_0\| < \delta \quad \Rightarrow \quad \lim_{\tau \to \tau_\infty} \|x(\tau)\| = 0,$$

  - where $\tau_\infty < \infty$ if the execution is Zeno and $\tau_\infty = \infty$ otherwise.

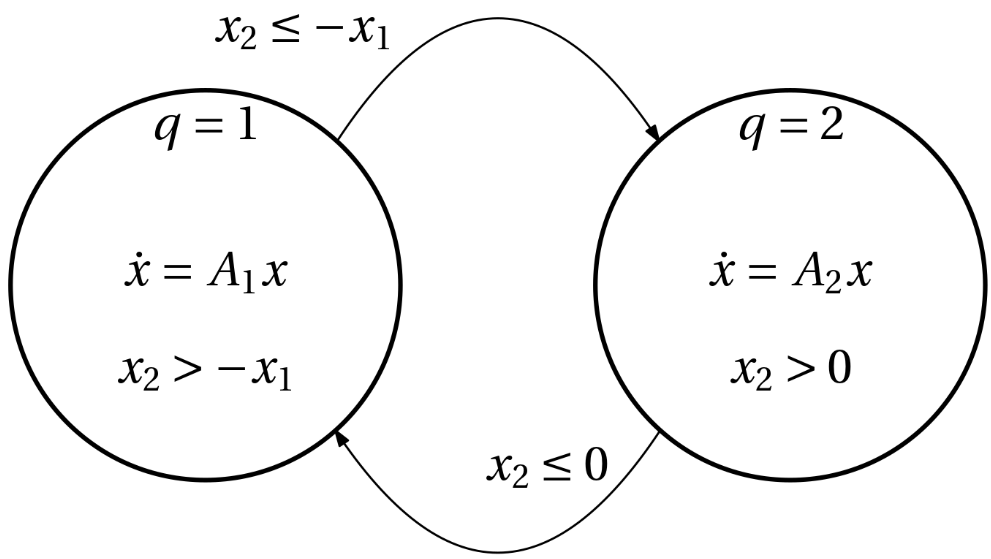
Must the individual dynamics be stable?



$$A_1 = \begin{bmatrix} -1 & -100 \\ 10 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}$$

- Both are stable.
- Switching can be destabilizing.

Can the individual dynamics be unstable?



$$A_1 = \begin{bmatrix} 1 & -100 \\ 10 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 10 \\ -100 & 1 \end{bmatrix}$$

- Both are unstable.
- Switching can be stabilizing.

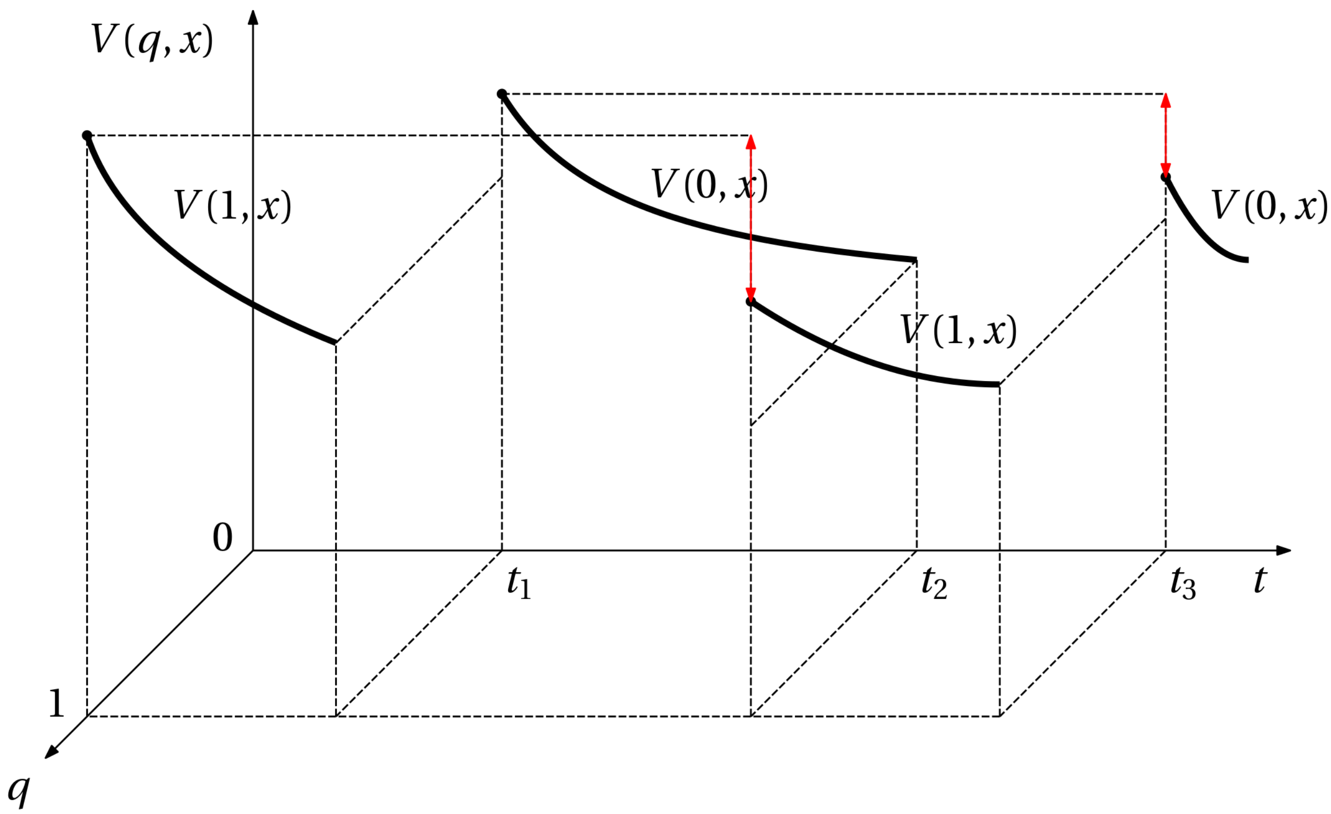
Hybrid system stability analysis via Lyapunov(-like) function

$V(q,x)$ smooth in x such that

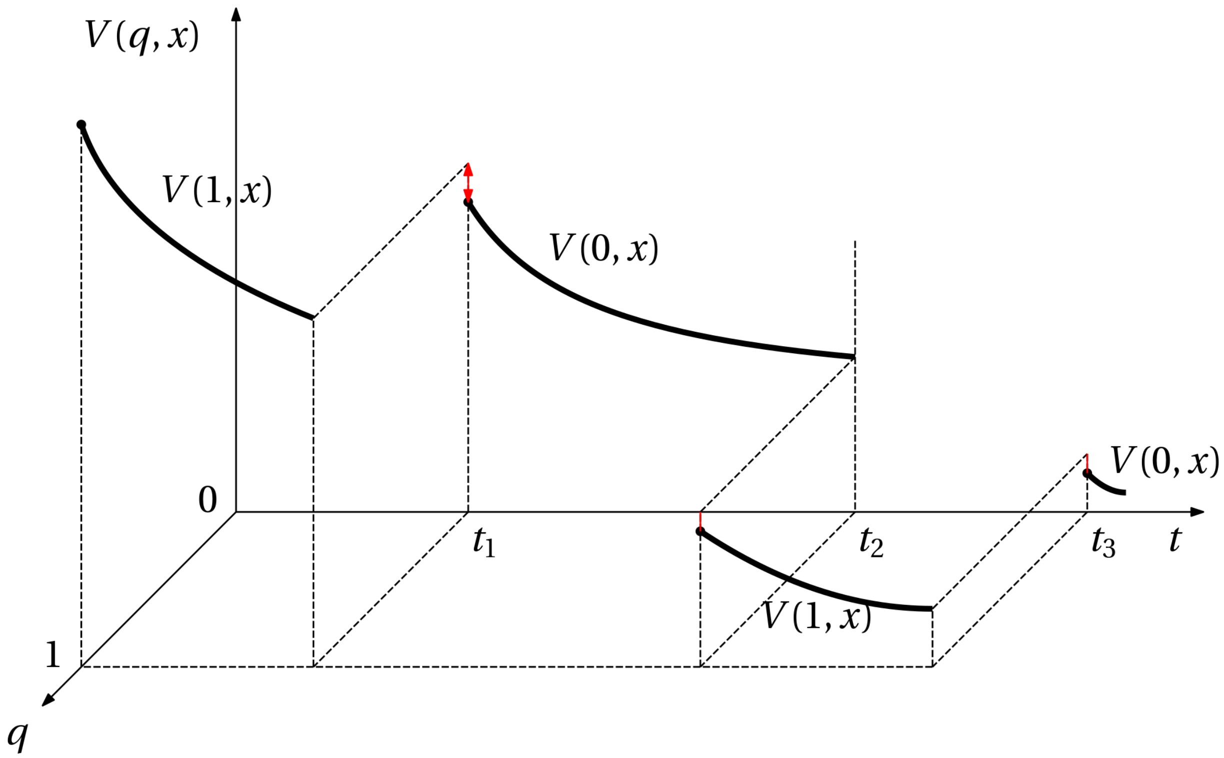
$$V(q,0) = 0, \quad V(q,x) > 0 \text{ for all nonzero } x,$$

$$\left(\nabla_x V(q,x)\right)^\top f(q,x) \leq 0$$

and



Stricter condition



Further restricted set of candidate functions: common Lyapunov function (CLF)

- Just a single Lyapunov function $V(x)$ common for all discrete states (modes, locations) q.
- Implies that arbitrary switching is allowed.
  - $\rightarrow$ conservative.
- For switched systems, the analysis can be interpreted within the framework of differential inclusions

$$\dot{x} \in \{f_1(x), f_2(x), \ldots, f_m(x)\}$$

(Global) uniform asymptotic stability

There exists $V(q)$ such that

$$\boxed{\kappa_1(\|x\|) \leq V(x) \leq \kappa_2(\|x\|),}$$

- where $\kappa_1(\cdot), \kappa_2(\cdot)$ are class $\mathcal{K}$ comparison functions,
- global for $\kappa_1(\cdot) \in \mathcal{K}_\infty$,

and

$$\boxed{\left(\nabla V(x)\right)^\top f_q(x) \leq -\rho(\|x\|), \quad q \in \mathcal{Q},}$$

where $\rho(\cdot)$ is a positive definite continuous function, zero at the origin.

Converse theorem for global uniform asymptotic stability (GUAS).

- Nice, a CLF exists for a GUAS system, but how do we find it?
- Restrict the set of candidate functions…

Common quadratic Lyapunov function (CQLF)

- A single quadratic Lyapunov function

$$V(x) = x^\top P x,$$

  where $P = P^\top \succ 0$.
- In principle could be used for nonlinear systems, but let's consider linear systems here.
- Consider r continuous-time LTI systems parameterized by the system matrices A_i for $i = 1, \ldots, r$ as

$$\dot{x} = A_i x(t).$$

- Time derivatives of $V(x)$ along the trajectory of the i-th system

$$\dot{V}(x) = x^\top (A_i^\top P + P A_i) x,$$

- which, upon introduction of new matrix variables $Q_i = Q_i^\top$ given by

$$A_i^\top P + P A_i = Q_i, \qquad i = 1, \ldots, r$$

- yields

$$\dot{V}(x) = x^\top Q_i x,$$

- from which it follows that **all** Q_i must satisfy

$$x^\top Q_i x \leq 0, \qquad i = 1, \ldots, r$$

for (Lyapunov) stability and

$$x^\top Q_i x < 0, \qquad i = 1, \ldots, r$$

for asymptotic stability.

Linear matrix inequality (LMI)

$$F(X) = F_0 + F_1 X G_1 + F_2 X G_2 + \ldots + F_k X G_k \succ 0$$

where the inequality reads "positive definite".

- Feasibility LMI problem: does $X = X^\top$ exist such that the LMI $F(X) \succ 0$ is satisfied?

CQLF as an LMI

$$
\begin{aligned}
P &\succ 0,\\
A_1^\top P + PA_1 &\prec 0,\\
A_2^\top P + PA_2 &\prec 0,\\
&\vdots\\
A_r^\top P + PA_r &\prec 0.
\end{aligned}
$$

Solving in Matlab using YALMIP or CVX

- Most numerical solvers for semidefinite programs (SDP) can handle nonstrict inequalities.
- Enforce strict inequality by

$$
\begin{aligned}
P &\succeq \epsilon I,\\
A_1^\top P + PA_1 &\preceq \epsilon I,\\
A_2^\top P + PA_2 &\preceq \epsilon I,\\
&\vdots\\
A_r^\top P + PA_r &\preceq \epsilon I.
\end{aligned}
$$

- For LMIs with no affine term, multiply them “arbitrarily” to get

$$
\begin{aligned}
P &\succeq I,\\
A_1^\top P + PA_1 &\preceq I,\\
A_2^\top P + PA_2 &\preceq I,\\
&\vdots\\
A_r^\top P + PA_r &\preceq I.
\end{aligned}
$$

Solution set of an LMI is convex

- and therefore if a solution $P = P^\top \succ 0$ exists such that

What if quadratic LF is not enough?

- Quadratic Lyapunov function is a (multivariate) polynomial

$$
\begin{aligned}
V(x) &= x^\top P x\\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12}\\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1\\ x_2 \end{bmatrix}\\
&= p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2
\end{aligned}
$$

- How about a polynomial of a higher degree?
- But how do we enforce positive definiteness?

Positive/nonnegative polynomials

- Is the polynomial $p(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^n$, positive (or nonnegative) on the whole $\mathbb{R}^n$? That is, we ask if

$$p(\boldsymbol{x}) > 0, \quad \text{or} \quad p(\boldsymbol{x}) \geq 0 \ \forall \boldsymbol{x} \in \mathbb{R}^n.$$

- Example: $p(\boldsymbol{x}) = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4 \geq 0$ for all $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}$?
- Additionally, $\boldsymbol{x}$ can be restricted to some $\mathcal{X} \subset \mathbb{R}^n$ and we ask if

$$p(\boldsymbol{x}) \geq 0 \ \forall \ \boldsymbol{x} \in \mathcal{X}.$$

- Semialgebraic sets $\mathcal{X}$ are often considered. These are defined by polynomial inequalities such as

$$g_j(\boldsymbol{x}) \geq 0,\ j = 1, \ldots, m.$$

How can we check nonnegativity of polynomials?

- Gridding is not the way to go – we need conditions on the coefficients of the polynomial so that we can do some optimization later.
- Example: Consider a univariate polynomial

  $$p(x) = x^4 - 4x^3 + 13x^2 - 18x + 17.$$

  Does it hold that $p(x) \geq 0 \; \forall x \in \mathbb{R}$?
- What if we learn that the polynomial can be written as

  $$p = (x-1)^2 + (x^2 - 2x + 4)^2$$

- Obviously, whatever the two squared polynomials are, after squaring they become nonnegative. And summing nonnegative numbers yields a nonnegative result. Let's generalized this.

Sum of squares (SOS) polynomials

- If we can express the polynomial as a sum of squares of some other polynomials, the original polynomial is nonnegative

$$\boxed{p(\boldsymbol{x}) = \sum_{i=1}^{k} p_i(\boldsymbol{x})^2 \;\Rightarrow\; p(\boldsymbol{x}) \geq 0,\; \forall \boldsymbol{x} \in \mathbb{R}^n.}$$

- But the converse does not hold in general – not every nonnegative polynomial is SOS!
- There are only three cases, for which SOS is a necessary and sufficient condition of nonnegativeness.
  - $n = 1$: univariate polynomials. The degree (the highest power) d can be arbitrarily high (but even, obviously).
  - $d = 2$ and n is arbitrary: multivariate polynomials of degree two. Example, $d = 3$ for $p(\boldsymbol{x}) = x_1^2 + x_1 x_2^2$.
  - $n = 2$ and $d = 4$: bivariate polynomials of degree 4 (at maximum).
- For all other cases all we can say is that

$$\mathrm{SOS} \Rightarrow p(\boldsymbol{x}) \geq 0.$$

How to get an SOS representation of a polynomial (or prove that none exist)?

- Back to the univariate example first.
- One of the two squared polynomials is x^2-2x+4.
- We can write it as

$$x^2-2x+4 = \underbrace{\begin{bmatrix}4 & -2 & 1\end{bmatrix}}_{\boldsymbol{v}^\top}\underbrace{\begin{bmatrix}1\\ x\\ x^2\end{bmatrix}}_{\boldsymbol{z}}.$$

- Then the squared polynomial can be written as

$$(x^2-2x+4)^2 = \boldsymbol{z}^\top\boldsymbol{v}\boldsymbol{v}^\top\boldsymbol{z}.$$

- Note that the the product $\boldsymbol{v}\boldsymbol{v}^\top$ is a positive semidefinite matrix of rank one.
- We can similarly express the second squared polynomial

$$x-1 = \underbrace{\begin{bmatrix}-1 & 1 & 0\end{bmatrix}}_{\boldsymbol{v}^\top}\underbrace{\begin{bmatrix}1\\ x\\ x^2\end{bmatrix}}_{\boldsymbol{z}}$$

and then

$$(x-1)^2 = \boldsymbol{z}^\top \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \boldsymbol{z}.$$

- Summing the two squares we get the original polynomial. But while doing this, we can sum the two rank-one matrices.

$$\begin{aligned} p(x) &= x^4 - 4x^3 + 13x^2 - 18x + 17 \\ &= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \boldsymbol{P} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \end{aligned},$$

where $\boldsymbol{P} \succeq 0$ is

$$\boldsymbol{P} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$$

- The matrix that defines the quadratic form is positive semidefinite and of rank 2.
  - The rank of the matrix is given by the number of squared terms in the SOS decomposition.
- In a general multivariate case we can proceed similarly. Just form the vector $\boldsymbol{z}$ from all possible monomials:

$$\begin{bmatrix} 1 \end{bmatrix}$$

$$z = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_1^2 \\ x_1 x_2 \\ \vdots \\ x_n^2 \\ \vdots \\ x_1 x_2 \ldots x_n^2 \\ \vdots \\ x_n^d \end{bmatrix}$$

But how to determine the coefficients of the matrix?

- Example: $p(x_1, x_2) = 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$.
- Define the vector $\boldsymbol{z}$ as

$$\boldsymbol{z} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}$$

- Then it must be possible to write the polynomial as

$$\begin{aligned} p(x_1, x_2) = &\begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}^\top \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix} \\ &= p_{11}x_1^4 + p_{33}x_2^4 \\ &\quad + 2p_{12}x_1^3x_2 + 2p_{23}x_1x_2^3 \\ &\quad + (2p_{13} + p_{22})x_1^2x_2^2 \end{aligned}$$

- This only gives 5 equations.
- The sixth is the LMI condition $P \succeq 0$.

Searching for a positive polynomial Lyapunov function

$$V(x) \quad \text{is SOS}$$

Ooops, $V(s)$ must be positive and not just nonnegative:

$$\boxed{V(s) - \phi(x) \quad \text{is SOS,}}$$

where $\phi(x) = \gamma \sum_{i=1}^{n} \sum_{j=1}^{d} x_i^{2j}$ for some $\gamma > 0$.

$$\boxed{\left(\nabla V(x)\right)^\top f(x) \quad \text{is SOS}}$$

Solving SOS problems

- Matlab: SOSTOOLS, …
- Julia: SumOfSquares.jl