Optimal control for continuous-time systems CHEATSHEET

Indirect approach via calculus of variations

1 Calculus of variations

Minimization over smooth enough (at least C^1) functions

$$\min_{y(x)\in\mathcal{C}^1[a,b]} J(y(x)).$$

Beware: x is the independent variable (often spatial position), $y()$ is a function.

Strong vs. *weak* minimum (easier theory for the weak).

Variation $\delta y()$ of a function (compare with $\mathrm{d}x$ in calculus)

$$y(x) = y^*(x) + \delta y(x)$$

Variation of a functional (=first-order approximation to ΔJ)

$$\delta J = \int_a^b \underbrace{\frac{\delta J}{\delta y(x)}}_{\text{variational derivative}} \delta y(x)\mathrm{d}x$$

First-order necessary condition of optimality

$$\delta J = 0, \quad \text{hence} \quad \frac{\delta J}{\delta y(x)} = 0$$

Recall in calculus $\mathrm{d}J = (\nabla J)^{\mathrm{T}}\mathrm{d}x = 0$, hence $\nabla J = 0$.

For the (cost) functional

$$J(y) = \int_a^b L(x,y,y')\mathrm{d}x$$

subject to

$$y(a) = y_a, \qquad y(a) = y_b,$$

the necessary conditions of optimality = *Euler-Lagrange equation*

$$\boxed{\frac{\partial L(x,y,y')}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L(x,y,y')}{\partial y'} = 0}$$

EL equation is a second-order ODE

$$\boxed{L_y - L_{y'x} - L_{y'y}y' - L_{y'y'}y'' = 0.}$$

hence two initial/boundary conditions needed.

Extensions for constrained problems (Lagrange multipliers).

Sufficiency difficult. Legandre condition

$$L_{y'y'} > 0$$

not sufficient. Jacobi condition needed (but not discussed).

2 General optimal control for a nonlinear system

$$\begin{aligned}\underset{\mathbf{x}(t),\mathbf{u}(t)}{\text{minimize}} \quad & \left[\phi(\mathbf{x}(t_\mathrm{f}),t_\mathrm{f}) + \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(\mathbf{x}(t),\mathbf{u}(t),t)\,\mathrm{d}t\right]\\ \text{subject to} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x},\mathbf{u},t)\\ & \mathbf{x}(t_\mathrm{i}) = \mathbf{r}_\mathrm{i}\\ & \mathbf{x}(t_\mathrm{f}) = \mathbf{r}_\mathrm{f} \text{ or } \mathbf{x}(t_\mathrm{f}) \text{ unspecified.}\end{aligned}$$

Hamiltonian (two conventions!)
as in physics

$$H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = \boldsymbol{\lambda}^{\mathrm{T}}(t)\,\mathbf{f}(\mathbf{x},\mathbf{u},t) - L(t,\mathbf{x},\mathbf{u}),$$

as favored in control theory

$$H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = L(t,\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}^{\mathrm{T}}(t)\,\mathbf{f}(\mathbf{x},\mathbf{u},t)$$

First-order necessary conditions—two-point boundary value problem (TP-BVP): *state* and *costate* ODEs plus *stationarity* equation.

$$\boxed{\begin{aligned}\mathbf{x}' &= \nabla_{\boldsymbol{\lambda}} H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda})\\ \boldsymbol{\lambda}' &= -\nabla_{\mathbf{x}} H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda})\\ \mathbf{0} &= \nabla_{\mathbf{u}} H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}).\end{aligned}}$$

If $\mathbf{u}$ eliminated, the resulting "true" canonical Hamiltonian ODEs solvable numerically (shooting, multiple shooting, collocation). In Matlab: `bvp4c` and `bvp5c`.

For "physics-convention" Hamiltonian

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{u},t)\\ \dot{\boldsymbol{\lambda}} &= \nabla_{\mathbf{x}} L - \nabla_{\mathbf{x}}\mathbf{f}\;\boldsymbol{\lambda},\\ \mathbf{0} &= \nabla_{\mathbf{u}} L - \nabla_{\mathbf{u}}\mathbf{f}\;\boldsymbol{\lambda}\\ \mathbf{x}(t_\mathrm{i}) &= \mathbf{r}_\mathrm{i}\\ \mathbf{x}(t_\mathrm{f}) &= \mathbf{r}_\mathrm{f} \text{ or } \boldsymbol{\lambda}(t_\mathrm{f}) = -\nabla\phi(\mathbf{x}(t_\mathrm{f}))\end{aligned}$$

For "control-convention" Hamiltonian

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{u},t)\\ \dot{\boldsymbol{\lambda}} &= -\nabla_{\mathbf{x}} L - \nabla_{\mathbf{x}}\mathbf{f}\;\boldsymbol{\lambda},\\ \mathbf{0} &= \nabla_{\mathbf{u}} L + \nabla_{\mathbf{u}}\mathbf{f}\;\boldsymbol{\lambda}\\ \mathbf{x}(t_\mathrm{i}) &= \mathbf{r}_\mathrm{i}\\ \mathbf{x}(t_\mathrm{f}) &= \mathbf{r}_\mathrm{f} \text{ or } \boldsymbol{\lambda}(t_\mathrm{f}) = \nabla\phi(\mathbf{x}(t_\mathrm{f}))\end{aligned}$$

3 LQ-optimal regulation on a finite time horizon

$$\begin{aligned}\underset{\mathbf{x}(t),\mathbf{u}(t)}{\text{minimize}} \quad & \Bigg[\mathbf{x}^{\mathrm{T}}(t_\mathbf{f})\underbrace{\mathbf{S}_\mathbf{f}}_{\geq 0}\mathbf{x}(t_\mathbf{f})\\ & + \int_0^{t_\mathrm{f}}\left(\mathbf{x}^{\mathrm{T}}(t)\underbrace{\mathbf{Q}}_{\geq 0}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\underbrace{\mathbf{R}}_{>0}u(t)\right)\mathrm{d}t\Bigg]\\ \text{subject to} \quad & \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\\ & \mathbf{x}(0) = \mathbf{r}_0,\\ & \mathbf{x}(t_\mathrm{f}) = \mathbf{r}_\mathrm{f} \text{ or } \mathbf{x}(t_\mathrm{f}) \text{ unspecified.}\end{aligned}$$

Could be also time-varying: $\mathbf{A}(t)$, $\mathbf{B}(t)$.

First-order necessary cond's ("physical Hamiltonian")

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\\ \dot{\boldsymbol{\lambda}} &= \mathbf{Q}\mathbf{x} - \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}\\ \mathbf{0} &= \mathbf{R}\mathbf{u} - \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}\\ \mathbf{x}(t_\mathrm{i}) &= \mathbf{r}_\mathrm{i}\\ \mathbf{x}(t_\mathrm{f}) &= \mathbf{r}_\mathrm{f} \text{ or } \boldsymbol{\lambda}(t_\mathrm{f}) = -\mathbf{S}_\mathrm{f}\mathbf{x}(t_\mathrm{f})\end{aligned}$$

Sufficiency (luckily) trivial: $\frac{\partial^2 J}{\partial u^2} = \mathbf{R} > 0$.

Thanks to nonsingularity of $\mathbf{R}$, from the stationarity equation express $\mathbf{u}$ and substitute to the state and costate equations to get Hamilton canonical equations

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\boldsymbol{\lambda}}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\\ \mathbf{Q} & -\mathbf{A}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \boldsymbol{\lambda}.\end{bmatrix}$$

For fixed-final state, the optimal control is a precomputed function/signal (the formula contains an inversion of weighted finite horizon controllability Gramian).

For free-final state, the optimal control is given by a state-feedback control

$$\boxed{\mathbf{u}(t) = -\underbrace{\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S}(t)}_{\mathbf{K}(t)}\mathbf{x}(t).}$$

where $\mathbf{S}(t) \geq 0$ is a positive semidefinite solution of (matrix) differential Riccati equation

$$\boxed{-\dot{\mathbf{S}} = \mathbf{S}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{Q} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S}.}$$

For long enough time horizon, $\mathbf{S}(t)$ (hence $\mathbf{K}(t)$) is most of time steady (constant).

4 LQR on an infinite time horizon

$$\begin{aligned}\underset{\mathbf{x}(t),\mathbf{u}(t)}{\text{minimize}} \quad & \int_0^\infty \left(\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)\right)\mathrm{d}t\\ \text{subject to} \quad & \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\\ & \mathbf{x}(0) = \mathbf{r}_0.\end{aligned}$$

Steady-state solution to the differential Riccati eqution found by solving Algebraic Riccati equation (ARE)

$$\boxed{\mathbf{0} = \mathbf{S}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{Q} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S}.} \qquad (1)$$

But ARE is a quadratic (matrix) equation and has more than one solution. We need a unique stabilizing $\mathbf{S} \geq 0$. Conditions:

- (A,B) *stabilizable*
- $(A,\sqrt{Q})$ *detectable*. If $(A,\sqrt{Q})$ *observable*, $\mathbf{S} > 0$ (nonzero feedback controller for stable systems).

LTI state-feedback

$$\boxed{\mathbf{u}(t) = -\underbrace{\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{S}}_{\mathbf{K}}\mathbf{x}(t).}$$

Solvers for ARE in Matlab (`icare`), Mathematica, Octave, Scilab, Julia, ...

Conservative robustness guarantees ($GM_+ = 2, GM_- = 1/2$, $PM = \pm 60°$).

5 LQ-optimal tracking, tracking+LQR

Optimal for an apriori known reference (not discussed). Tracking for class of references (steps, ramps, ...): proportional feedforward, integral control, state augmentation.