Limits of achievable performance for feedback control systems

Graduate course on Optimal and Robust Control (spring'22)

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Lecture outline

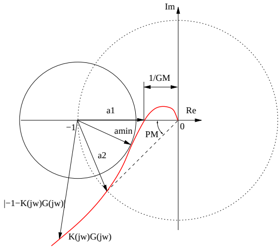
SISO systems

- Requirements on performance expressed in frequency domain
- Ideal controller
- Limits on S and T
- Interpolation conditions on internal stability
- Limits given by unstable poles and zeros
- Limits given by presence of delay
- Limits given by presence of disturbances and references
- Limits due to saturation of inputs

MIMO systems

- Directions in MIMO systems
- Conditioning of a system
- Relative Gain Array (RGA)
- Interpolation conditions of internal stability and related limitations
- Limits given by the input constraints (saturation)
- Limits given by uncertainty in the model

Magnitude and phase margins vs. sensitivity function



$$\alpha_{min} = \inf_{\omega} |-1 - K(j\omega)G(j\omega)| = \frac{1}{\|S(s)\|_\infty}$$

$$GM = \frac{1}{1-\alpha_1}; \quad PM = 2\arcsin\frac{\alpha_2}{2}$$

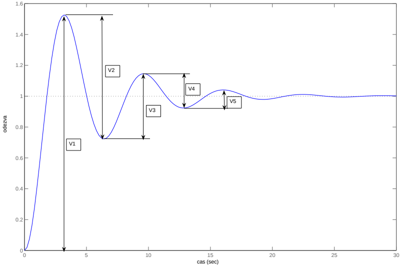
$$GM \geq \frac{M_S}{M_S - 1}; \quad PM \geq 2\arcsin\frac{1}{2M_S}$$

For commonly required values $GM = 2$ a $PM = 30°$:
$M_S = 2(6dB)$.

Peaks in time vs. peaks in frequency domain

Total variations

$$TV = \sum_{i=1}^{\infty} v_i$$



$$\|T\|_\infty \le TV \le (2n+1)\|T\|_\infty$$

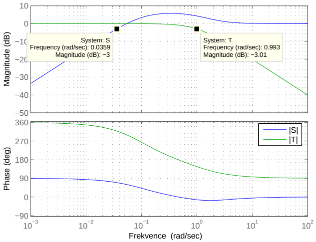
$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \omega_n = 1 rad/s, \ \zeta = 0.2$$

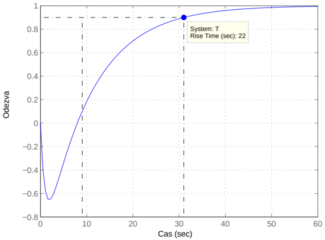
$$TV = 3.22 \ M_T = 2.55, \ M_S = 2.73$$

Bandwidth

S vs. T

$$L(s) = \frac{-s + 0.1}{s(s + 2.1)}$$





Signal interpretation of $\mathcal{H}_\infty$ norm

1. harmonic input (sin, cos)
2. general input (finite energy)

Ideal controller with inverse dynamics

$$y(s) = G(s)u(s) + G_d(s)d(s)$$

Desired $r = y$, hence in open-loop $u = G^{-1}r - G^{-1}G_d d$. For feedback controller $u = K(r - y)$

$$\begin{aligned} u &= K(r - Gu - G_d d) = Kr - KGu - KG_d d \\ &= (I + KG)^{-1}Kr - (I + KG)^{-1}KG_d d = SKr - SKG_d d \\ &= G^{-1}GSKr - G^{-1}GSKG_d d = G^{-1}Tr - G^{-1}TG_d d \end{aligned}$$

For frequencies where $T \approx I$, ideal controller includes an inverse, but need not be realizable, if

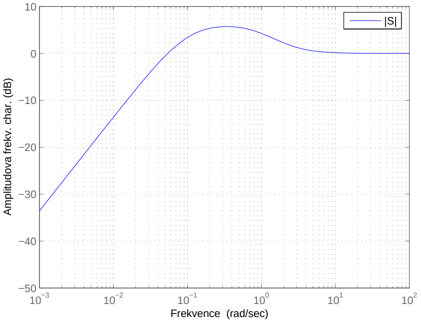
- G includes RHP zeros, its inverse is unstable,
- G includes delays, its inverse contains predictions,
- G includes more finite poles then zeros, its inverse not realizable,
- inverse of an uncertain G not precisely realizable in open-loop,
- $|G^{-1}R|$ is large ($\gg 1$ for scaled model),
- $|G^{-1}G_d|$ is large,

$S + T = 1$

$$S = \frac{1}{1+L}, \quad T = \frac{L}{1+L}$$

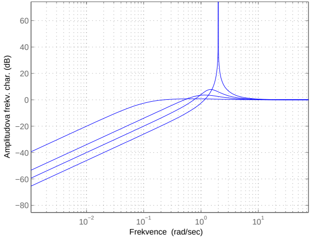
Waterbed effect: for relative degree at least 2

$$\int_{\omega=0}^{\infty} \ln |S(j\omega)| d\omega = \pi \sum_{i=1}^{N_p} \Re(p_i)$$



Waterbed effect: for RHP zero

$$L(s) = \frac{k}{s}\frac{2-s}{2+s}, \quad k = 0.1, 0.5, 1.0, 2.0$$



$$\int_{\omega=0}^{\infty} \ln|S(j\omega)| w(z,\omega) d\omega = \pi \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{\bar{p}_i - z} \right|$$

for one real zero: $w(z,\omega) = \frac{2z}{z^2+\omega^2} = \frac{2}{z}\frac{1}{1+(\omega/z)^2}$

for a complex pair: $w(z,\omega) = \frac{x}{x^2+(y-\omega)^2} + \frac{x}{x^2+(y+\omega)^2}$

Interpolation conditions on internal stability

$$\boxed{T(p) = 1, \quad S(p) = 0 \quad S(z) = 1, \quad T(z) = 0}$$

Maximum modulus principle

$$\|f(s)\|_\infty = \sup_\omega |f(j\omega)| \geq |f(s_0)| \quad \forall s_0 \in \text{RHP}$$

G has an "unstable" zero:

$$\|W_p S\|_\infty \geq |W_p(z)|$$

G has unstable pole

$$\|WT\|_\infty \geq |W(p)|$$

$G(s)$ has N_p unstable poles p_i and N_z unstable zeros z_j

$$\|W_p S\|_\infty \geq c_{1j} |W_p(z_j)|, \quad c_{1j} = \prod_{i=1}^{N_p} \frac{|z_j + \bar{p}_i|}{|z_j - p_i|} \geq 1$$

$$\|WT\|_\infty \geq c_{2i} |W(p_i)|, \quad c_{2i} = \prod_{j=1}^{N_z} \frac{|\bar{z}_j + p_i|}{|z_j - p_i|} \geq 1$$

Limits given by unstable poles and zeros

$$\|S\|_\infty > c, \quad \|T\|_\infty > c, \quad c = \frac{|z+p|}{|z-p|}$$

For $G(s) = \frac{s-4}{(s-1)(0.1s+1)}$ at best:

$$\|S\|_\infty > 1.67$$

$$\|T\|_\infty > 1.67$$

Interpolation conditions: constraints on bandwidth

Two requirements:

$$|S(j\omega)| < \frac{1}{|W_p(j\omega)|} \quad \forall\omega \quad \iff \|W_pS\|_\infty < 1$$

$$\|W_pS\|_\infty \geq |W_p(z)|$$

Hence at least

$$|W_p(z)| < 1$$

$$|W_p(z)| = \left|\frac{s/M + \omega_B^*}{s + \omega_B^* A}\right|$$

One real “unstable“ zero:

$$\omega_B^*(1-A) < z\left(1 - \frac{1}{M}\right)$$

$A = 0$ a $M = 2$: at least $\omega_B^* < 0.5z$.
For complex conjugate pair

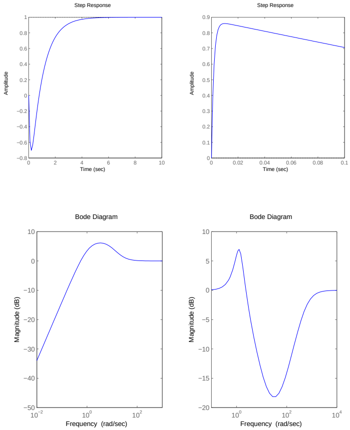
$$\omega_B^* = |z|\sqrt{1 - \frac{1}{M^2}}$$

$M = 2$: $\omega_B^* < 0.86|z|$.

Constraints at low vs. high frequencies

$$G(s) = \frac{-s+1}{s+1}$$

$$K_1(s) = K_c \frac{s+1}{s} \frac{1}{0.05s+1}, \quad K_2(s) = -K_c \frac{s}{(0.05s+1)(0.02s+1)}$$



Constraints given by unstable poles

Using

$$|T(j\omega)| < \frac{1}{|W(j\omega)|} \quad \forall\omega \quad \iff \|WT\|_\infty < 1$$

and the interpolation condition $\|WT\|_\infty \geq |W(p)|$:

$$|W(p)| < 1$$

With weight

$$W(s) = \frac{s}{\omega_{BT}^*} + \frac{1}{M_T}$$

we get a lower bound on the bandwidth

$$\omega_{BT}^* > p\frac{M_T}{M_T - 1}$$

$M_T = 2$: $\omega_{BT}^* > 2p$
For complex conjugate pair: $\omega_{BT}^* > 1.15|p|$.

Limits given by presence of delay

Ideal transfer function

$$T(s) = e^{-\theta s}$$

Unit gain over all frequencies, but phase decaying linearly.

$$S(s) = 1 - e^{-\theta s} \approx \theta s$$

Gain 1 at about $\omega_c = 1/\theta$.

$$\boxed{\omega_c < \frac{1}{\theta}}$$

Limits given by presence of disturbances and references

1. $d(t) = \sin(\omega t)$
2. necessary scaling
3. reference as a special disturbance $d(t) = -Rr(t)$
4. we require $|e| = |G_d d| < 1$

$$e(s) = S(s)G_d(s)d(s)$$

The requirement of an acceptably small regulation error implies the constraint

$$|S(j\omega)G_d(j\omega)| < 1 \quad \forall\omega \iff \|SG_d\|_\infty < 1$$

If the system has unstable zero, we get from from interpolation conditions of internal stability the constraint

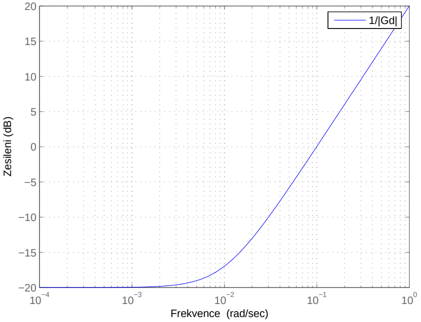
$$|G_d(z)| < 1$$

$$\boxed{\omega_B > \omega_d}$$

where ω_d is frequency where $|G_d(j\omega)| = 1$.

Limits given by presence of disturbances and references

$G_d(s) = k_d/(1 + \tau_d s)$, kde $k_d = 10$ a $\tau_d = 100$, (model already scaled)



Bandwidth at least 0.1 rad/s (or even more).

Limits due to saturation of inputs

Bound $u(t) \leq 1$

$$u = G^{-1}r - G^{-1}G_d d$$

Condition to avoid saturation

$$|G^{-1}(j\omega)G_d(j\omega)| < 1 \quad \forall\omega$$

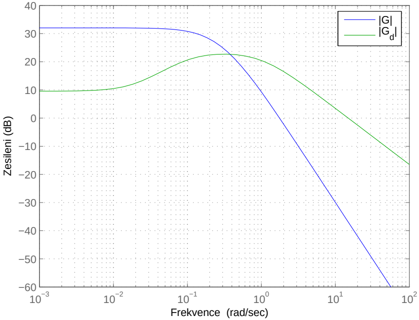
Conflict for unstable systems

$$\boxed{|G(j\omega)| > |G_d(j\omega)| \quad \forall\omega < 2p}$$

Limits due to saturation of inputs

Example

$$G(s) = \frac{40}{(5s+1)(2.5s+1)}, \quad G_d(s) = \frac{50s+1}{(10s+1)(s+1)}$$

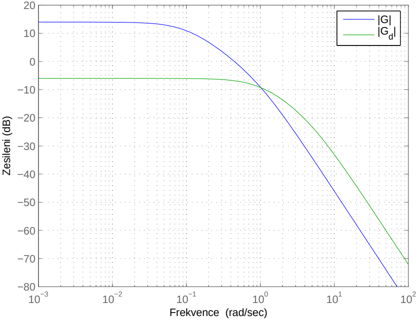


For disturbances at frequencies between 0.38 rad/s and 15 rad/s can appear saturation. Do not need to consider higher frequencies (no control there).

Limits due to saturation of inputs

Example

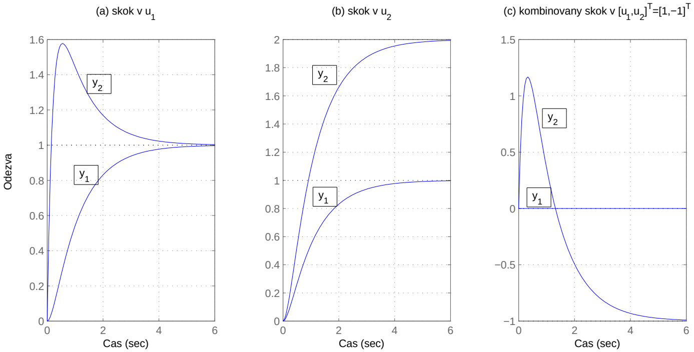
$$G(s) = \frac{5}{(10s+1)(s-1)}, \quad G_d(s) = \frac{k_d}{(s+1)(0.2s+1)}, \quad k_d < 1$$



Not due to disturbance, but due to unstable pole up to 2rad/s. For $k_d = 0.5$ is a problem.

Directions in MIMO systems

$$G(s) = \frac{1}{(0.2s+1)(s+1)} \begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

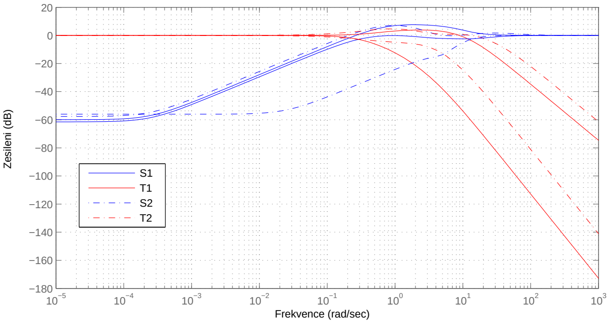


Zero of G at 0.5.

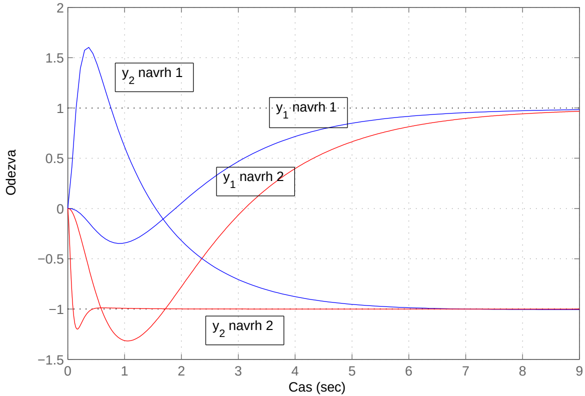
Shifting the influence of RHP zero into another channel

$$\min_K \left\| \begin{bmatrix} W_p S \\ W_u KS \\ WT \end{bmatrix} \right\|_\infty$$

1. $M_1 = M_2 = 1.5; \quad \omega_{B1}^* = \omega_{B2}^* = z/2 = 0.25$
2. $M_1 = M_2 = 1.5; \quad \omega_{B1}^* = z/2; \omega_{B2}^* = 25$, (higher demands on the second channel).



Shifting the influence of RHP zero into another channel



Input and output directions, directions of poles and zeros

For control the ouput variants a bit more useful:

- y_z: output direction of an RHP zero (constant),
- y_p: output direction of an unstable pole (constant),
- $y_d(s)$: output direction of a disturbance (frequency dependent),
- $u_i(s)$: $i-$th output direction (left singular vector) of the system (frequency dependent).

Conditioning of a system

$$\boxed{\gamma(G) = \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)}}$$

1. Ill-conditioned for $\gamma > 10$
2. But depends on scaling!

Therefore minimized conditioning number

$$\boxed{\gamma^*(G) = \min_{D_1, D_2} \gamma(D_1 G D_2)}$$

but difficult do compute (=upper bound on μ)
RGA can be used to give a reasonable estimate.

Relative Gain Array (RGA) as an indicator of difficulties with control

$$\boxed{\Lambda(G) = G \circ (G^{-1})^T}$$

- independent of scaling,
- sum of elements in rows and columns is 1,
- sum of absolute values of elements of RGA is very close to the minimized sensitivity number γ^*, hence a system with large RGA entries is always ill-conditioned (but system with large γ can have small RGA),
- RGA for a triangular system is an identity matrix,
- relative uncertainty of an element of a transfer function matrix equal to (negative) inverse of the corresponding RGA entry makes the system singular.

Example: RGA, conditioning number, minimized conditioning number

$$G = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$$

Should the troubles with control be expected?

Functional controllability

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{2}{s+2} & \frac{4}{s+2} \end{bmatrix}$$

$$y_o(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad y_o(\infty) = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Interpolation conditions: zeros and zero directions

T and S distinguished not only by poles and zeros of G, but also by their directions.

$$\boxed{y_z^H T(z) = 0; \quad y_z^H S(z) = y_z^H}$$

$$\boxed{S(p) y_p = 0; \quad T(p) y_p = y_p}$$

Remark: the above are for output sensitivity and complementary sensitivity functions.

Bandwidth limitations due to unstable poles and zeros

Similarly as for SISO systems

$$\|W_p S\|_\infty = \sup_{j\omega} |W_p(j\omega)| \bar{\sigma}(S(j\omega)) \geq |W_p(z)|$$

$\omega_B^* < z/2$ or $\omega_B^* > 2z$. This constraint holds for the worst direction only.

$$\|WT\|_\infty \geq |W(p)|$$

$\omega_B^* > 2|p|$ again for the worst direction.
Combination

$$\|S\|_\infty \geq c; \quad \|T\|_\infty \geq c; \quad c = \sqrt{\sin^2\phi + \frac{|z+p|^2}{|z-p|^2} cos^2 \phi}$$

where $\phi = \arccos|y_z^H y_p|$ is the angle between the directions of the pole and the zero.

Limits given by presence of disturbance and/or reference

Consider single disturbance at a time, model g_d (to avoid worst-case effect). Its effect at the output is $y = g_d d$. Disturbance direction is

$$y_d = \frac{1}{\|g_d\|_2} g_d$$

Disturbance condition number is

$$\gamma_d(G) = \bar{\sigma}(G)\bar{\sigma}(G^\dagger y_d)$$

Varies between 1 for the disturbance direction aligned with $\bar{u}$ and $\gamma(G)$ when it is aligned with $\underline{u}$

Limits given by presence of disturbance and/or reference

Assumed scaled model: worst-case disturbance is $|d(j\omega)| = 1 \forall\omega$ and the error expected $\|e(j\omega)\|_2 < 1 \forall\omega$ (whether 2-norm or ∞-norm... does not matter).
With feedback control $e = Sg_d d$:

$$\|Sg_d\|_2 = \bar{\sigma}(Sg_d) < 1 \;\; \forall\omega \quad \Longleftrightarrow \quad \|Sg_d\|_\infty < 1$$

Equivalent to

$$\boxed{\|Sy_d\|_2 < \frac{1}{\|g_d\|_2} \;\; \forall\omega}$$

Hence sensitivity S must be less then $1/\|g_d\|_2$ **in the disturbance direction y_d only.**

$$\underline{\sigma}(S)\|g_d\|_2 \leq \|Sg_d\|_2 \leq \bar{\sigma}(S)\|g_d\|_2$$

At least $\bar{\sigma}(I+L) > \|g_d\|_2$ and perhaps $\underline{\sigma}(I+L) > \|g_d\|_2$.

Disturbance rejection by a plant with RHP zero

Consider the interpolation condition $y_z^H S(z) = y_z^H$ and apply *maximum modulus theorem* to $f(s) = y_z^H S g_d$

$$\|S g_d\|_\infty \geq |y_z^H g_d(z)| = |y_z^H y_d| \|g_d\|_2$$

To safisfy $\|S g_d\|_\infty < 1$ must have at least

$$|y_z^H g_d(z)| < 1$$

(which is a generalization of $G_d(z) < 1$ for SISO systems).

Example: Disturbance rejection by a plant with RHP zero

Consider the system

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}, \quad g_d(s) = \frac{6}{s+2}\begin{bmatrix} k \\ 1 \end{bmatrix}, \quad |k| \leq 1$$

Is it possible to get $\|Sg_d\|_\infty < 1$ for arbitrary $|k| \leq 1$? The RHP zero is at $s = 4$.

```
>> G = sdf([s-1, 4; 4.5, 2*(s-1)],(s+2))
G =
     -1 + s       4
      4.5        -2 + 2s
    ------------------------
           2 + s

>> roots(G)
ans =
     4

>> Gz = value(G,4)
Gz =
     0.5000     0.6667
     0.7500     1.0000

>> [U,S,V] = svd(Gz)
U =
    -0.5547    -0.8321
    -0.8321     0.5547
S =
     1.5023          0
          0     0.0000
V =
    -0.6000    -0.8000
    -0.8000     0.6000

>> Uz = U(:,2)
Uz =
    -0.8321
     0.5547
```

Then condition for attenuation of disturbances is then

$$\left| \begin{bmatrix} -0.8321 & 0.5547 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix} \right| < 1$$

The **necessary** condition is then $k > -0.54$. (Other factor as well, input saturation, uncertainty, ...)

Limits given by the input constraints (saturation)

Similar reasoning as with disturbances, see the book (page 240 and 241).

Limits given by uncertainty in the model: in open loop

Distinguish uncertainty at the input and output

$$G = G_0(I + E_I) \quad \text{or} \quad E_I = G_0^{-1}(G - G_0)$$

$$G = (I + E_O)G_0 \quad \text{or} \quad E_O = (G - G_0)G_0^{-1}$$

Open-loop control

$$e_0 = y_0 - r = G_0 u - r = G_0 \underbrace{G_0^{-1} r}_{K} - r$$

With uncertainty at the output

$$e = GG_0^{-1} r - r = (GG_0^{-1} - 1)r = E_O r$$

With uncertainty at the input

$$e = G_0 E_I G_0^{-1} r$$

For instance, for diagonal uncertainty at the input
$[G_0 E_I G_0^{-1}]_{ii} = \sum_{j=1}^{n} \lambda_{ij}(G_0)\epsilon_j$

Limits given by uncertainty in the model: in feedback

What changes by introduction of feedback?

$$
\begin{aligned}
S &= (I + GK)^{-1} \\
&= (I + (I + E_O)G_0K)^{-1} \\
&= \left((I + E_O \underbrace{G_0K(I + G_0K)^{-1}}_{T_0})(I + G_0K) \right)^{-1} \\
&= S_0(I + E_O T_0)^{-1}
\end{aligned}
$$

The difference between the nominal and uncertain sensitivity function is

$$S_0 - S = T - T_0 = SE_O T_0$$

hence the deviation of regulation error

$$e - e_0 = -Sr - (-S_0 r) = (S_0 - S)r = SE_O T_0 r$$

Upper bound on $\bar{\sigma}(S)$ for uncertainty at the output

$$\bar{\sigma}(S) \leq \bar{\sigma}(S_0)\bar{\sigma}((I + E_O T_0)^{-1}) \leq \frac{\bar{\sigma}(S_0)}{1 - |W_O|\bar{\sigma}(T_0)}$$

Upper bound on $\bar{\sigma}(S)$ for uncertainty at the input

First

$$\begin{aligned} S &= S_0(I + G_0 E_I G_0^{-1} T_0)^{-1} = S_0 G_0 (I + E_I T_{0I})^{-1} G_0^{-1} \\ &= (I + T_0 K^{-1} E_I K)^{-1} S_0 = K^{-1}(I + T_{0I} E_I)^{-1} K S \end{aligned}$$

where T_{0I} is input complementary sensitivity for the nominal system G_0

$$T_{0I} = K G_0 (I + K G_0)^{-1}$$

For arbitrary or diagonal uncertainty and arbitrary controller

$$\begin{aligned} \bar{\sigma}(S) &\leq \gamma(G_0)\bar{\sigma}(S_0)\bar{\sigma}((I + E_I T_{0I})^{-1}) \leq \gamma(G_0)\frac{\bar{\sigma}(S_0)}{1 - |W_I|\bar{\sigma}(T_{0I})} \\ \bar{\sigma}(S) &\leq \gamma(K)\bar{\sigma}(S_0)\bar{\sigma}((I + T_{0I} E_I)^{-1}) \leq \gamma(K)\frac{\bar{\sigma}(S_0)}{1 - |W_I|\bar{\sigma}(T_{0I})} \end{aligned}$$

Findings: ill-conditioned system can be very nonrobust for uncertainty at the input. But can be conservative.

Decoupling controllers

Considering controllers $K(s) = l(s)G^{-1}(s)$. Then $S = sI$ and $T = tI$ and input uncertainty at each input w_I. Then there is a model from the full family such that

$$\bar{\sigma}(S) \geq \bar{\sigma}(S_0)\left(1 + \frac{|w_I t_0|}{1 + |w_I t_0|}\|\Lambda(G_0)\|_{i\infty}\right)$$

where $\|\Lambda(G_0)\|_{i\infty}$ is row-sum-abs norm and $\bar{\sigma}(S_0)$ is $|s_0|$. (For proof see the book, page 249)

Example: diagonal controller for ill-conditioned system

Consider a simplified model of a distillation column in LV configuration

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$$

```
>> G = sdf([87.8, -86.4; 108.2, -109.6],(75*s+1))
G =
     88             -86
     1.1e+02        -1.1e+02
    ---------------------
            1 + 75s

>> G_0 = value(G,0)
G_0 =
    87.8000   -86.4000
   108.2000  -109.6000

>> RGA = G_0.*inv(G_0).'
RGA =
    35.0688   -34.0688
   -34.0688    35.0688
```

and hence

$$G_0 E_I G_0^{-1} = \begin{bmatrix} 35.0688\epsilon_1 - 34.0688\epsilon_2 & * \\ * & -34.0688\epsilon_1 + 35.0688\epsilon_2 \end{bmatrix}$$

and largest diagonal elements for opposite signs of ϵ_i. For 20% input uncertainty

```
>> E_I = diag([0.2, -0.2])
E_I =
    0.2000         0
         0   -0.2000
>> G_0*E_I*inv(G_0)
ans =
   13.8275  -11.0582
   17.2868  -13.8275
```

Theferore feedback needed. But even feedback will have troubles

```
>> gam = cond(G_0)
gam =
  141.7320

>> sum(sum(abs(RGA)))
ans =
  138.2752
```