Stability of hybrid systems

Multiple Lyapunov function approach

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November 2023

Example: CQLF cannot be found

Multiple Lyapunov Function (MLF) approach to analysis of stability

- Not just a single common Lyapunov function $V(\boldsymbol{x})$.
- Instead, several Lyapunov-like functions $V_i(\boldsymbol{x}),\ i = 1, \ldots, r$, that are Lyapunov on some subsets of the state space Ω_i.
- Stitch them together to form

$$V(x) = \begin{cases} V_1(x) & \text{if } x \in \Omega_1, \\ \vdots \\ V_r(x) & \text{if } x \in \Omega_r, \end{cases}$$

S-procedure

- A result about solvability of two or more quadratic inequalities, not necessarily convex ones.
    - For convex problems we have nonlinear Farkas' lemma.
- Origin in the control theory (analysis of stability of nonlinear systems, hence the S letter) with the first rigorous result provided by Yakubovich in 1970s.
- It gives conditions under which (satisfaction of) one quadratic inequality follows from (satisfaction of) another one (or more), that is, it gives a condition under which the following implication holds:
- Quadratic inequality #1 satisfied by some $x \Rightarrow$ Quadratic inequality #0 satisfied by the same x.
- In other words, it gives a condition under which the solution set of the inequality #1 denoted as $\mathcal{X}_1$ is included in the solution set $\mathcal{X}_0$ of the inequality #0.

S-procedure with nonstrict inequalities

- Consider the general quadratic functions

$$F_i(x) = x^\top A_i x + 2b_i^\top x + c_i,\ i = 0, \ldots, p.$$

- Question: under which conditions $F_0(x) \geq 0$ for all x satisfying $F_i(x) \geq 0,\ i = 1, \ldots, p$?
- In other words, we are looking for conditions under which the implication

$$F_i(x) \geq 0,\ i = 1, \ldots, p \quad \Rightarrow \quad F_0(x) \geq 0$$

holds.

- In the simplest (yet relevant) case $n = 1$ we search for conditions under which $F_0(x) \geq 0$ for all x satisfying $F_1(x) \geq 0$,
- that is, conditions under which the implication

$$F_1(x) \geq 0 \quad \Rightarrow \quad F_0(x) \geq 0$$

holds.

Sufficient conditions

The existence of $\alpha_i \geq 0,\ i = 1, \ldots, p$ such that

$$F_0(x) - \sum_{i=1}^{p} \alpha_i F_i(x) \geq 0.$$

- Generally not necessary, the condition is conservative.
- It can be formulated as an LMI

$$\begin{bmatrix} A_0 & b_0 \\ b_0^\top & c_0 \end{bmatrix} - \sum_{i=1}^{p} \alpha_i \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \succeq 0$$

where $\alpha_i \geq 0,\ i = 1, \ldots, p$.

Sufficient and also necessary

- It is nontrivial that for $p=1$ it is also necessary, provided that there is some x_0 such that $F_1(x_0)>0$.
- Then we have the following equivalence between the two constraints:

$$
\begin{aligned}
F_0(x)\geq 0 \;\forall x \text{ satisfying } F_1(x)\geq 0\\
\Longleftrightarrow\\
F_0(x)-\alpha F_1(x)\geq 0,\; \text{for some } \alpha\in\mathbb{R},\; \alpha\geq 0,
\end{aligned}
$$

- which again can be formulated as an LMI, namely

$$
\begin{bmatrix}A_0 & b_0\\ b_0^\top & c_0\end{bmatrix}-\alpha\begin{bmatrix}A_1 & b_1\\ b_1^\top & c_1\end{bmatrix}\succeq 0,\quad \alpha\geq 0.
$$

More on S-procedure

- There are several variants
  - strict, nonstrict or mixed inequalities,
  - just two or more,
  - some of the constraints can be equations.
- Literature (freely available online)
  - Boyd, S., El Ghaoui, L., Feron, E., Balakrishnan, V., 1994. Linear Matrix Inequalities in System and Control Theory. SIAM. Pages 23 and 24.
  - Boyd, S., Vandenberghe, L., 2004. Convex Optimization. Cambridge University Press. Page 655.

Piecewise quadratic Lyapunov function

- Quadratic Lyapunov-like functions, that is, quadratic functions $V_i(\boldsymbol{x}) = \boldsymbol{x}^\top \mathbf{P}_i \boldsymbol{x}$ that qualify as Lyapunov functions only on respective subsets $\boldsymbol{\Omega}_i \subset \mathbb{R}^n$:

$$V_i(\boldsymbol{x}) = \boldsymbol{x}^\top \mathbf{P}_i \boldsymbol{x} > 0 \quad \forall\, \boldsymbol{x} \in \boldsymbol{\Omega}_i$$

$$\dot{V}_i(\boldsymbol{x}) = \boldsymbol{x}^\top \left(\mathbf{A}_i^\top \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_i\right) \boldsymbol{x} < 0 \quad \forall\, \boldsymbol{x} \in \boldsymbol{\Omega}_i$$

Using comparison functions and nonstrict inequalities

$$\alpha_1 \boldsymbol{x}^\top \mathbf{I} \boldsymbol{x} \leq \boldsymbol{x}^\top \mathbf{P}_i \boldsymbol{x} \leq \alpha_2 \boldsymbol{x}^\top \mathbf{I} \boldsymbol{x} \quad \forall\, \boldsymbol{x} \in \Omega_i$$

$$\boldsymbol{x}^\top \left(\mathbf{A}_i^\top \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_i\right) \boldsymbol{x} \leq -\alpha_3 \boldsymbol{x}^\top \mathbf{I} \boldsymbol{x} \quad \forall\, \boldsymbol{x} \in \Omega_i$$

Characterization of subsets of state space using LMI

- Some subsets $\Omega_i \subset \mathbb{R}^n$ characterized using linear and quadratic inequalities can be formulated as within the LMI framework as

$$\boldsymbol{x}^\top \mathbf{Q}_i \boldsymbol{x} \geq 0.$$

- In particular, centered ellipsoids and cones.
- For example,

$$\begin{aligned}\Omega_i = \{\boldsymbol{x} \in \mathbb{R}^n \mid (\mathbf{c}^\top \boldsymbol{x} \geq 0 \land \mathbf{d}^\top \boldsymbol{x} \geq 0)\\ \lor (\mathbf{c}^\top \boldsymbol{x} \leq 0 \land \mathbf{d}^\top \boldsymbol{x} \leq 0)\}.\end{aligned}$$

- This constraint can be reformulated as

$$(\mathbf{c}^\top \boldsymbol{x})(\mathbf{d}^\top \boldsymbol{x}) \geq 0,$$

Combining the subset characterization and Lyapunov-ness using the S-procedure

$$\alpha_i \boldsymbol{x}^\top \mathbf{I} \boldsymbol{x} \leq \boldsymbol{x}^\top \mathbf{P}_i \boldsymbol{x},$$

$$\boldsymbol{x}^\top \left(\mathbf{A}_i^\top \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_i\right) \boldsymbol{x} \leq -\gamma_i \boldsymbol{x}^\top \mathbf{I} \boldsymbol{x},$$

both for all $\boldsymbol{x} \in \Omega_i$, that is, all $\boldsymbol{x}$ such that $\boldsymbol{x}^\top \mathbf{Q}_i \boldsymbol{x} \geq 0$.

$$\mathbf{P}_i - \alpha_i \mathbf{I} - \mu_i \mathbf{Q}_i \succeq 0, \quad \mu_i \geq 0,\ \alpha_i > 0,$$

$$\mathbf{A}_i^\top \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_i + \gamma_i \mathbf{I} + \xi_i \mathbf{Q} \preceq 0, \quad \mu_i \geq 0,\ \gamma_i > 0.$$

More general sets

$$\boldsymbol{x}^\top \mathbf{Q}\boldsymbol{x} + 2\mathbf{r}^\top \boldsymbol{x} + s \geq 0$$

$$\begin{bmatrix}\boldsymbol{x}^\top & 1\end{bmatrix} \underbrace{\begin{bmatrix}\mathbf{Q} & \mathbf{r}\\ \mathbf{r}^\top & s\end{bmatrix}}_{\bar{\mathbf{Q}}} \underbrace{\begin{bmatrix}\boldsymbol{x}\\ 1\end{bmatrix}}_{\bar{\boldsymbol{x}}} \geq 0$$

$$\begin{bmatrix}\mathbf{Q} & \mathbf{r}\\ \mathbf{r}^\top & s\end{bmatrix} \succeq 0$$

Example: affine subspace

$$\mathbf{c}^\top \boldsymbol{x} + d \geq 0$$

$$\begin{bmatrix}\boldsymbol{x}^\top & 1\end{bmatrix} \begin{bmatrix}\mathbf{0} & \mathbf{c}\\ \mathbf{c}^\top & 2d\end{bmatrix} \begin{bmatrix}\boldsymbol{x}\\ 1\end{bmatrix} \geq 0$$

$$\begin{bmatrix}\mathbf{0} & \mathbf{c}\\ \mathbf{c}^\top & 2d\end{bmatrix} \succeq 0$$

But then the Lyapunov-like functions and system matrices must also be extended

$$V(\boldsymbol{x}) = \begin{bmatrix}\boldsymbol{x}^\top & 1\end{bmatrix} \underbrace{\begin{bmatrix}\mathbf{P} & \mathbf{P}_{12}\\ \mathbf{P}_{12} & P_{22}\end{bmatrix}}_{\bar{\mathbf{P}}} \begin{bmatrix}\boldsymbol{x}\\ 1\end{bmatrix}$$

$$\bar{\mathbf{A}} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & 0\end{bmatrix}$$