Complementarity systems

Yet another framework for hybrid systems

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Complementarity constraints

- Two optimization variables (possibly vectors) x and y satisfy the complementarity constraint if x or y is equal to zero and both are nonnegative:

$$xy = 0,\ x \geq 0,\ x \geq 0,$$

- or, using a dedicated compact notation:

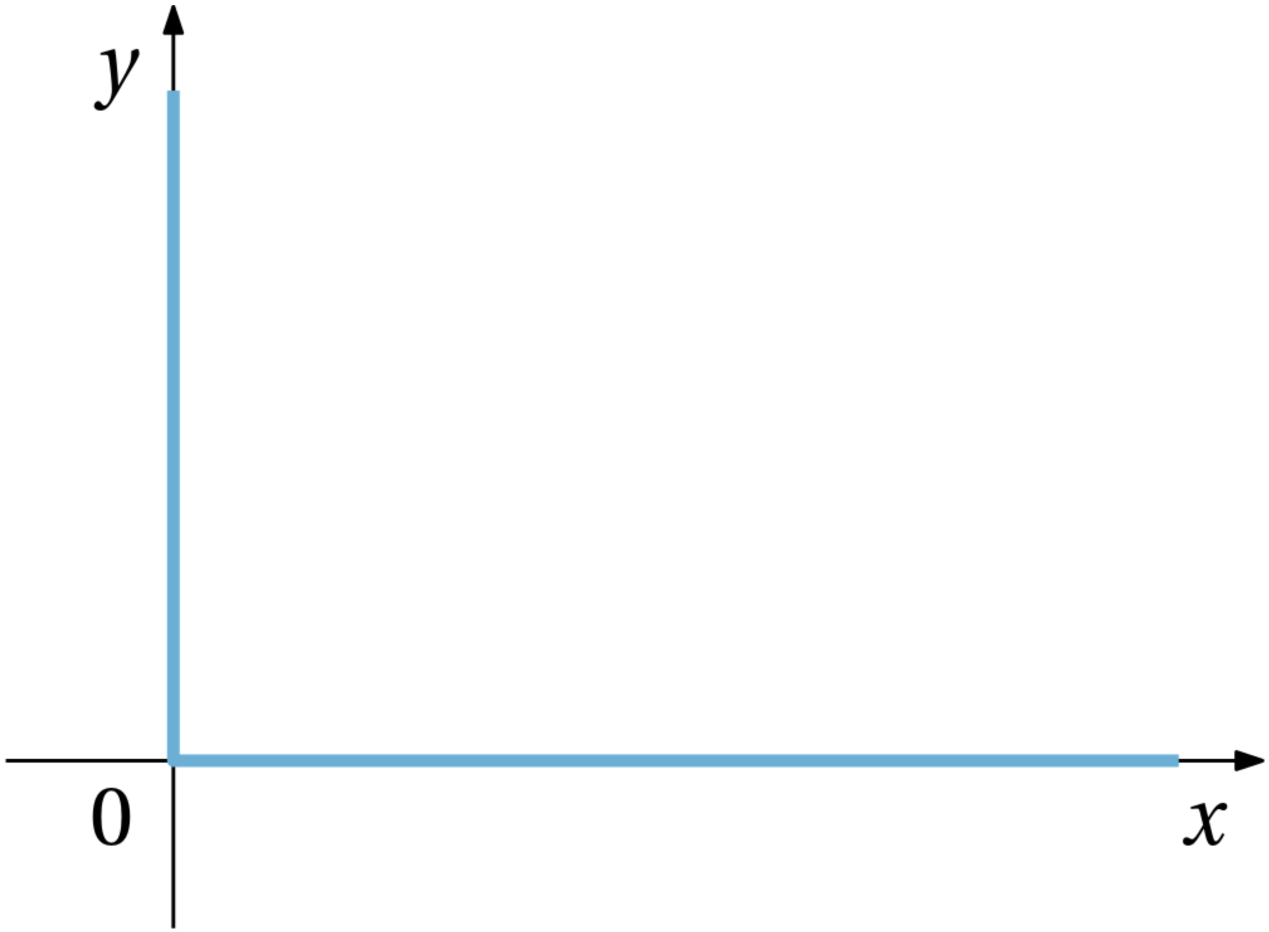
$$\boxed{0 \leq x \perp y \geq 0.}$$

- Inequalities interpreted componentwise for vectors

$$\boxed{\mathbf{0} \leq \boldsymbol{x} \perp \boldsymbol{y} \geq \mathbf{0}.}$$

Geometric interpretation of complementarity constraints

- The set of admissible pairs (x, y) in the $\mathbb{R}^2$ plane is constrained to the L-shaped subset given by the nonnegative x and y semi-axes (including the origin).
  - $\Rightarrow$ nonconvex set.



- Optimization over these constraints is usually difficult, conditions for *Constraint qualification* are not met.

Linear complementarity problem (LCP)

- For a given square matrix $\mathbf{M}$ and a vector $\mathbf{q}$, the linear complementarity problem asks for finding two vectors $\boldsymbol{w}$ and $\boldsymbol{z}$ satisfying

$$\boldsymbol{w} - \mathbf{M}\boldsymbol{z} = \mathbf{q}$$

$$\mathbf{0} \leq \boldsymbol{w} \perp \boldsymbol{z} \geq \mathbf{0}.$$

- Just by moving all the provided data to the right hand side

$$\boldsymbol{w} = \underbrace{\mathbf{M}\boldsymbol{z} + \mathbf{q}}_{\mathbf{f}(\boldsymbol{z})}$$

$$\mathbf{0} \leq \mathbf{f}(\boldsymbol{z}) \perp \boldsymbol{z} \geq \mathbf{0},$$

Existence of a unique solution

of every vector $\mathbf{q}$ if and only if the matrix $\mathbf{M}$ is a P-matrix.

LCP related to LP and QP

because KKT conditions conditions come in the form of LCP.

- Consider the QP problem with inequality constraints

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\boldsymbol{x}^\top \mathbf{Q}\boldsymbol{x} + \mathbf{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \mathbf{A}\boldsymbol{x} \geq \mathbf{b}, \quad \boldsymbol{x} \geq \mathbf{0}, \end{aligned}$$

- The KKT conditions are (compare to the conditions for the equality-constrained QP) can be written

$$\begin{aligned} \mathbf{0} \leq \boldsymbol{x} \perp \mathbf{Q}\boldsymbol{x} - \mathbf{A}^\top \boldsymbol{\lambda} + \mathbf{c} \geq \mathbf{0} \\ \mathbf{0} \leq \boldsymbol{\lambda} \perp \mathbf{A}\boldsymbol{x} - \mathbf{b} \geq \mathbf{0}. \end{aligned}$$

- These can be reformatted as

Nonlinear complementarity problem

Given a vector function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, find a vector $\boldsymbol{x} \in \mathbb{R}^n$ satisfying

$$\mathbf{0} \leq \boldsymbol{x} \perp \mathbf{f}(\boldsymbol{x}) \geq \mathbf{0}.$$

Mixed complementarity problem (MCP)

- Extension of complementarity constraint to the situation in which the variable x is lower- and upper-bounded. In particular, it can be stated as

$$l \leq x \leq u \perp f(x).$$

- The convention for interpretation is
  - If x is strictly within the interval, that is, $l < x < u$, then $f(x) = 0$,
  - If $x = l$, then $f(x) \geq 0$,
  - if $x = u$, then $f(x) \leq 0$.

Extended linear complementarity problem (ELCP)

- Given some matrices $\mathbf{A}$ and $\mathbf{B}$, vectors $\mathbf{c}$ and $\mathbf{d}$, and m subsets $\phi_j \subset \{1, 2, \ldots, p\}$, find a vector $\boldsymbol{x}$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (\mathbf{A}\boldsymbol{x} - \mathbf{c})_i = 0,$$

$$\mathbf{A}\boldsymbol{x} \geq \mathbf{c},$$

$$\mathbf{B}\boldsymbol{x} = \mathbf{d},$$

  or show that no such $\boldsymbol{x}$ exists.

- The first equation is equivalent to

Mathematical programm with equilibrium constraints (MPCC)

- The mathematical program with complementarity constraints (MPCC) is

$$
\begin{aligned}
\operatorname*{minimize}_{\boldsymbol{x}\in\mathbb{R}^n}\ & f(\boldsymbol{x})\\
\text{subject to } & 0 \leq h(\boldsymbol{x}) \perp g(\boldsymbol{x}) \geq 0.
\end{aligned}
$$

- Special case of *Mathematical program with equilibrium constraints (MPEC)*.

Mathematical program with equilibrium constraints (MPEC)

- Optimization problem in which some variable should satisfy equilibrium constraints:

$$\min_{x_1, x_2} f(x_1, x_2)$$
$$\text{subject to } \nabla_{x_2} \phi(x_1, x_2) = 0$$

- For convex $\phi()$ it can be reformulated into a *Bilevel optimization* problem.

Bilevel optimization

- Optimization problem in which some variables are constrained to be results of some inner optimization.
- In the simplest form

$$
\begin{aligned}
&\min_{x_1, x_2} f(x_1, x_2) \\
&\text{s. t.} \;\; x_2 = \arg\min_{x_2} \; \phi(x_1, x_2)
\end{aligned}
$$

Disjunctive constraints

- A number of affine constraints combined with $\vee$ and $\wedge$ logical operators.

$$T_1 \vee T_2 \vee \ldots \vee T_m,$$

where

$$T_i = T_{i1} \wedge T_{i1} \wedge \ldots \wedge T_{in_i},$$

where

$$T_{ij} = c_{ij} x + d_{ij} \in \mathcal{D}_{ij}.$$

Software for solving complementarity problems

- PATH solver https://pages.cs.wisc.edu/~ferris/path.html.
- Julia (JuMP package)

```
julia> @variable(model, x >= 0)
x

julia> @constraint(model, 2x - 1 ⊥ x)
[2 x - 1, x] ∈ MathOptInterface.Complements(2)
```

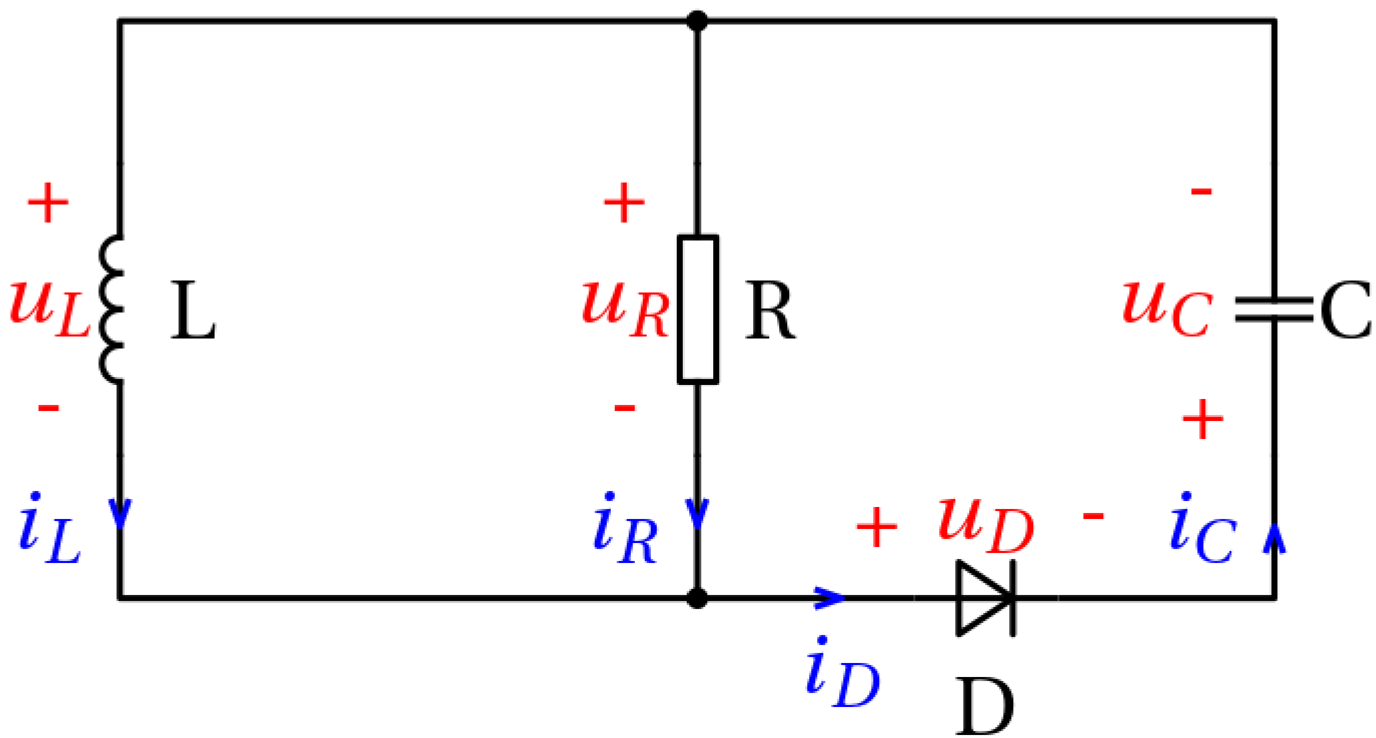
- Matlab
  - Optimization Toolbox for Matlab: `fmincon` can only handle it as a general nonlinear constraint $g_1(x)g_2(x)$.
  - Complementarity constraints in Tomlab.
  - Complementarity constraints in Knitro (Artelys).
  - YALMIP: through the command complements: `F = complements(w >= 0, z >= 0)`.

Linear complementarity system (LCS)

- Also: Linear dynamical complementarity problem (LDCP).
- It extends a classical linear dynamical system with linear complementarity constraints.

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t)\\
y(t) &= Cx(t) + Du(t)\\
0 \leq u(t) &\perp y(t) \geq 0.
\end{aligned}
$$

Example: Electrical circuit with a diode as an LCS



- Note the upside-down orientation of the voltage and the current for the capacitor – we wanted the diode current identical to the capacitor current.
- Following the charge formalism within Lagrangian modelling, we choose the generalized coordinates as

$$q = \begin{bmatrix} q_L \\ q_C \end{bmatrix}.$$

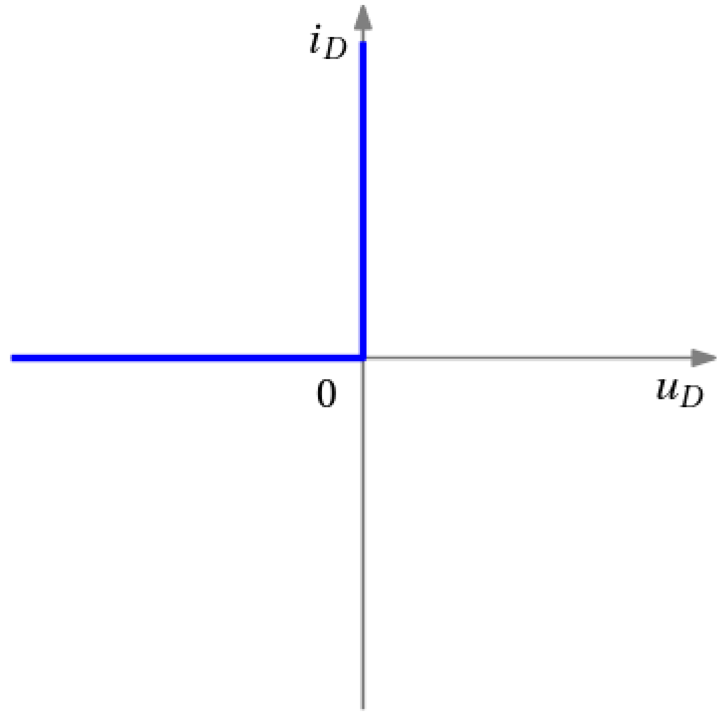
- That this is indeed a sufficient number is obvious, but we can also check the classical formula $B - N + 1 = 4 - 3 + 1 = 2$.
- We can choose the state variables as

$$x = \begin{bmatrix} i_L \\ q_c \end{bmatrix}.$$

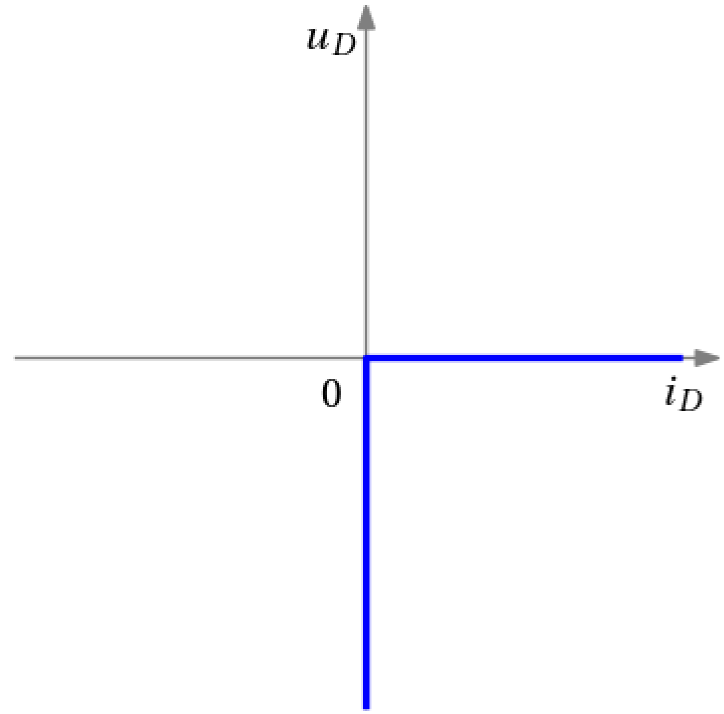
- The resulting state equations are

$$\begin{aligned} i_L' &= -\frac{1}{LC} q_C - \frac{1}{L} u_D \\ q_C' &= i_L - \frac{1}{RC} q_C - \frac{1}{R} u_D. \end{aligned}$$

- The idealized volt-ampere characteristics of the diode is



- Flipping the axes to get the current as the horizontal axis



- And finally after introducing an auxiliary variable (in fact, the reverse voltage of the diode) $\bar{u}_D = -u_D$, we get the desired dependence

Example continued: Combining the state equations with the diode complementarity constraint

(upon replacing the diode voltage with its reverse $\bar{u}_D$ while using $i_D = i_C$), we get

$$
\begin{aligned}
i_L' &= -\frac{1}{LC} q_C + \frac{1}{L} \bar{u}_D \\
q_C' &= i_L - \frac{1}{RC} q_C + \frac{1}{R} \bar{u}_D \\
0 &\leq q_C' \perp \bar{u}_D \geq 0.
\end{aligned}
$$

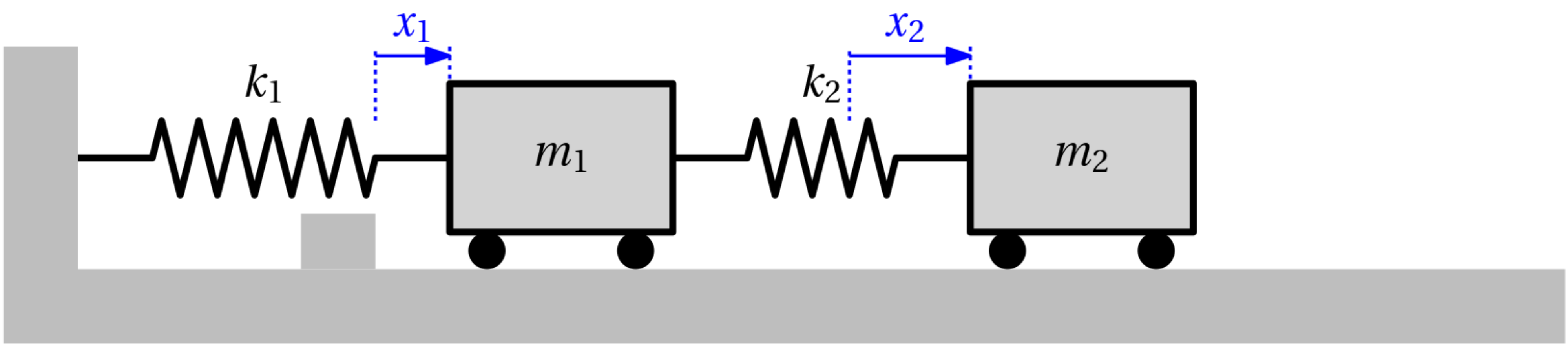
- We are not there yet – there is a derivative in the complementarity constraint. But just substitute for it:

$$
\begin{aligned}
i_L' &= -\frac{1}{LC} q_C + \frac{1}{L} \bar{u}_D\\
q_C' &= i_L - \frac{1}{RC} q_C + \frac{1}{R} \bar{u}_D\\
0 &\leq i_L - \frac{1}{RC} q_C + \frac{1}{R} \bar{u}_D \perp \bar{u}_D \geq 0.
\end{aligned}
$$

and voila, we finally got the LCS description.

- We can also reformat it into the vector format

Mass-spring system with a hard stop as a linear complementarity system



- Two carts moving horitontally (left or right) interconnected through a spring.
- The left cart also interconnected with the wall through a another spring.
- Furthemore, the motion of the left cart is constrained in that there is a hard stop that prevents the cart from moving further to the left.
- The variables x_1 and x_2 give deviations of the two carts from their equilibrium positions. The hard stop is located at such equilibrium position of the left cart.
- Besides the two positions, their derivatives are also introduced as state vectors.
- The input u corresponds to the reaction force of the hard

The input u corresponds to the reaction force of the hard stop.

- As the output, only the position of the left cart is (arbitrarily) chosen.

$$
\begin{aligned}
\dot{x}_1(t) &= x_3 \\
\dot{x}_2(t) &= x_4 \\
\dot{x}_3(t) &= -\frac{k_1 + k_2}{m_1} x_1(t) + \frac{k_2}{m_1} x_2(t) + \frac{1}{m_1} u(t) \\
\dot{x}_4(t) &= \frac{k_2}{m_2} x_1(t) - \frac{k_2}{m_2} x_2(t) \\
y(t) &= x_1(t)
\end{aligned}
$$

- The presence of the hard stop can be modelled as an inequality constraint on the state (or the output in this case)

$$x_1(t) = y(t) \geq 0.$$

- Strictly speaking, similar constraint should also be imposed on the right cart. That one can not overcome the hard stop either. Furthermore, the left cart would stand in the way too. But we ignore it here.
- The reaction force u can only be nonnegative

$$u(t) \geq 0.$$

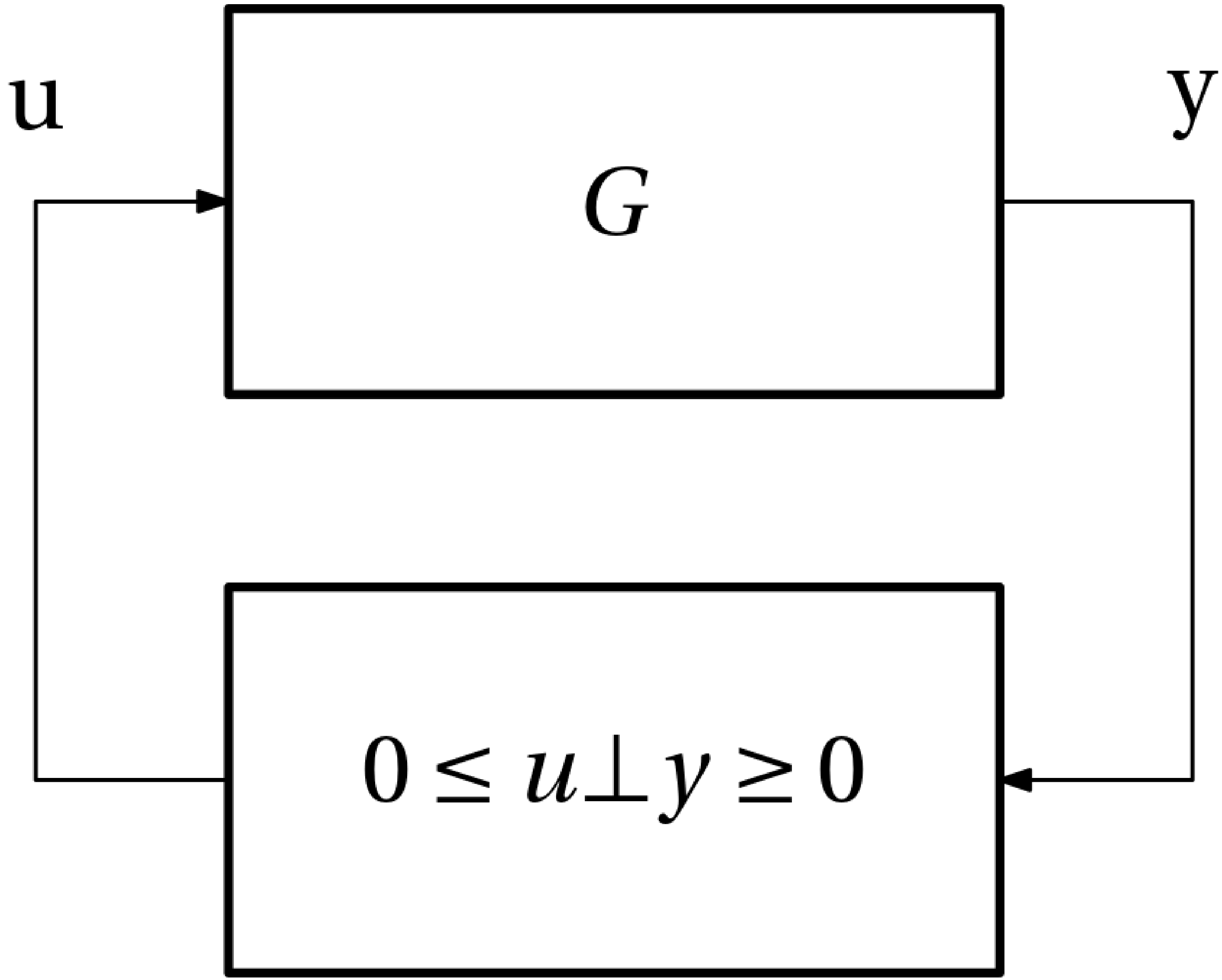
- Furthermore, the reaction force is acting if and only if the left cart hits the hard stop, that is cart hits the hard stop, that is,

$$y(t)u(t) = 0.$$

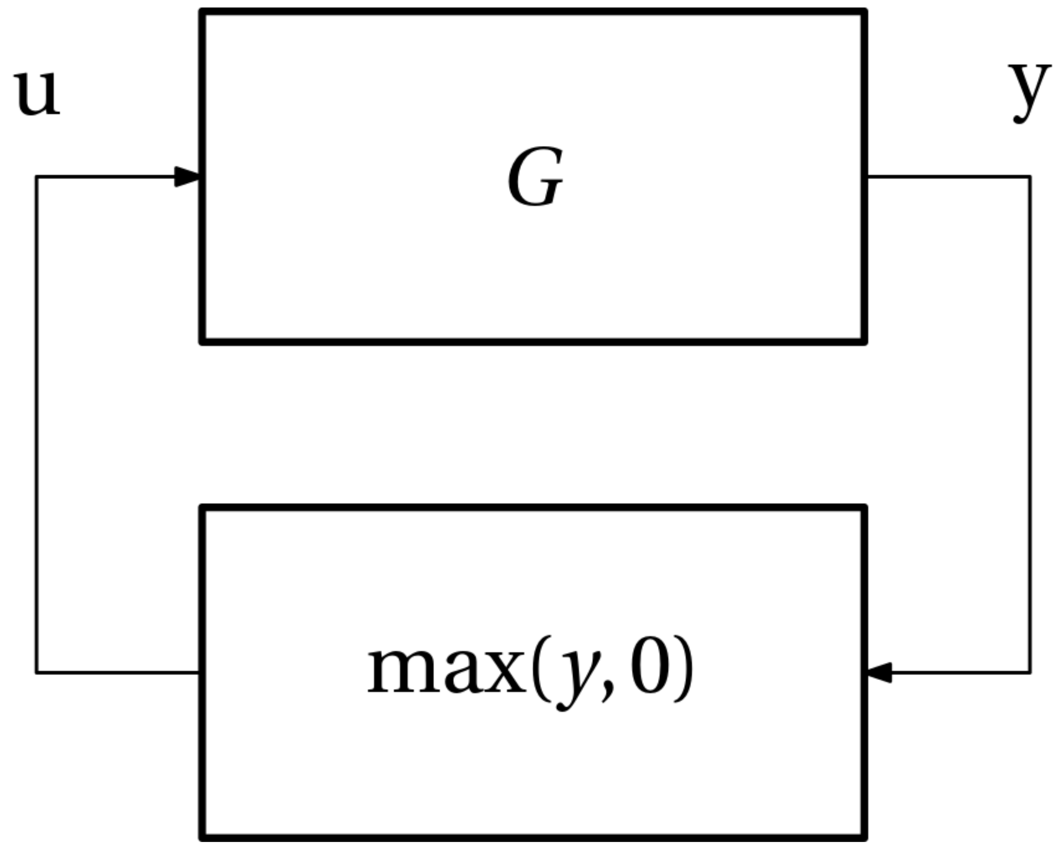
- All the above three constraints can be written compactly as a complementarity constraing

$$0 \leq y(t) \perp u(t) \geq 0.$$

Complementarity system as a feedback interconnection


u
y
G
0 ≤ u⊥y ≥ 0

Complementarity systems vs PWA and max-plus linear systems



$$y = y^+ - y^-, \quad 0 \leq y^+ \perp y^- \geq 0.$$

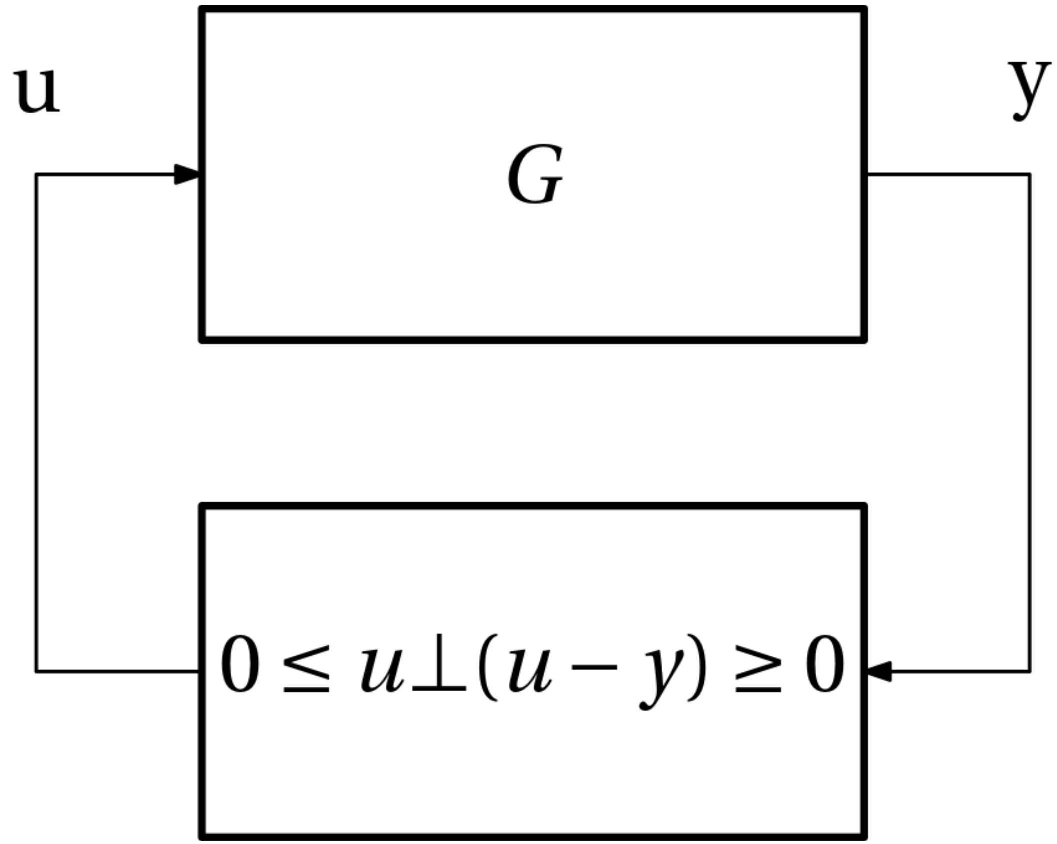
$$\max(y, 0) = \max(y^+ - y^-, 0) = y^+.$$

- Set

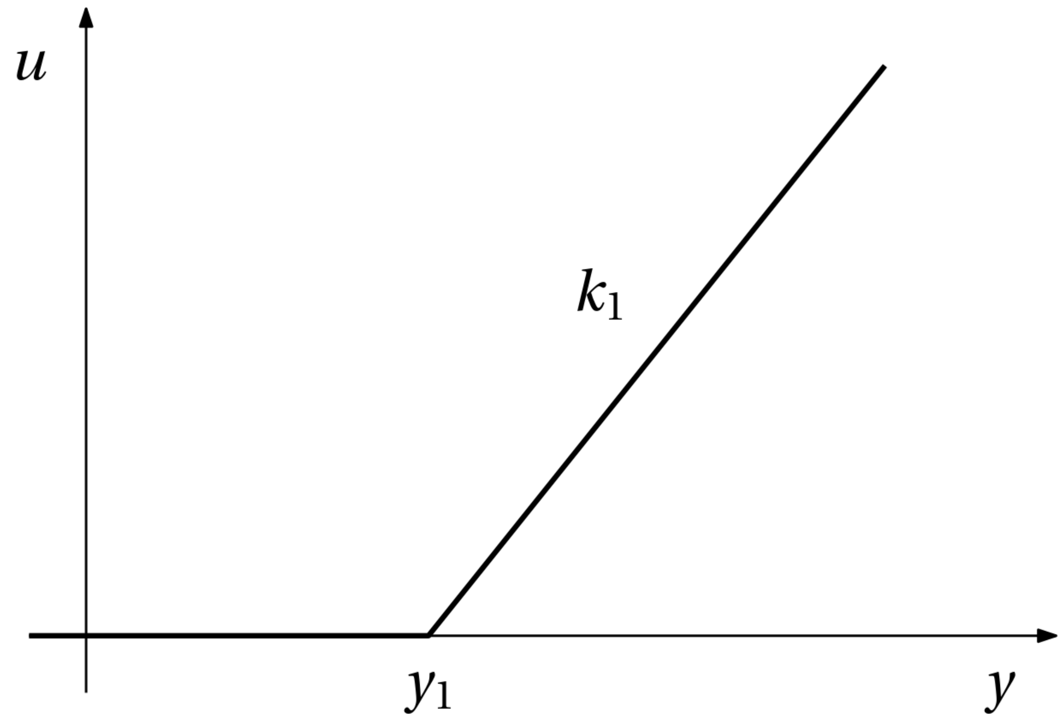
$$y^+ = u$$

- then

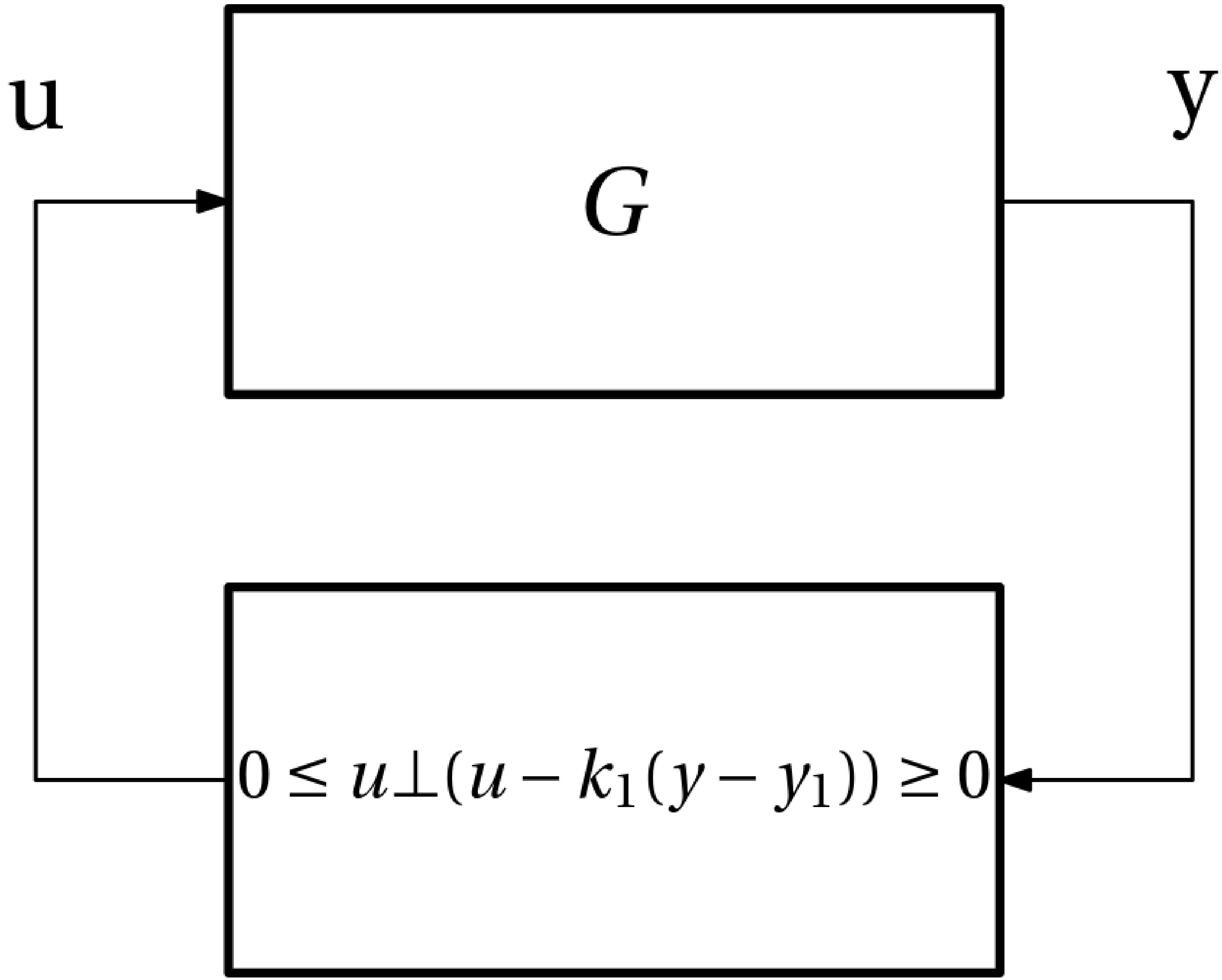
$$y = u - y^- \qquad y^- = u - y$$


u
y
G
$0 \leq u \perp (u - y) \geq 0$

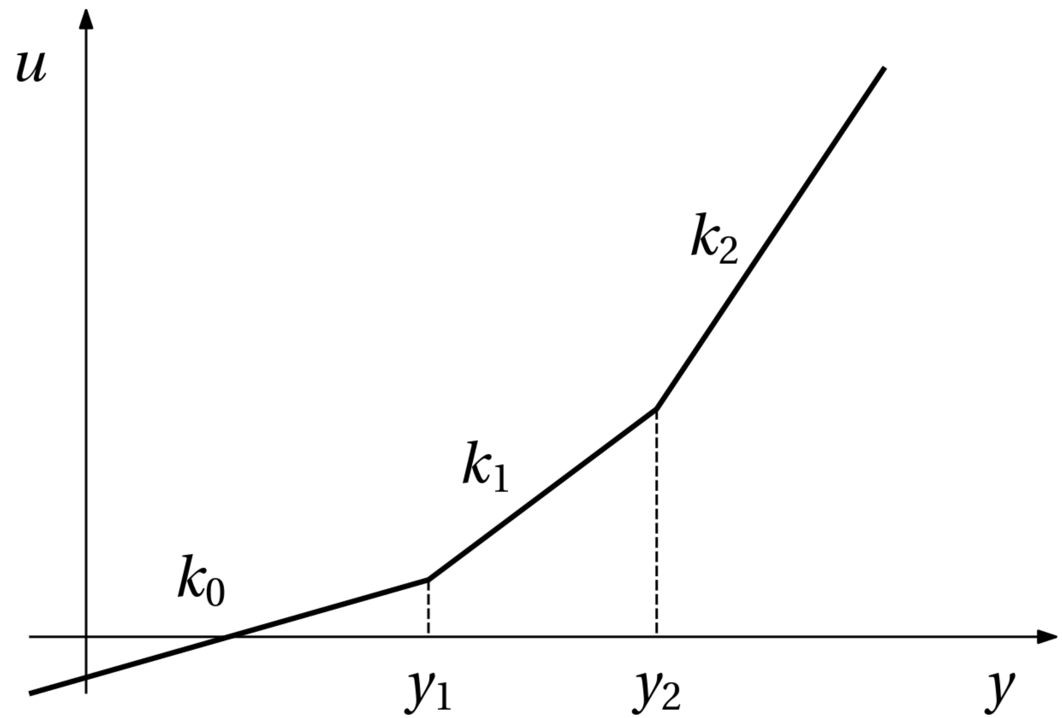
More complicated PWA functions in feedback



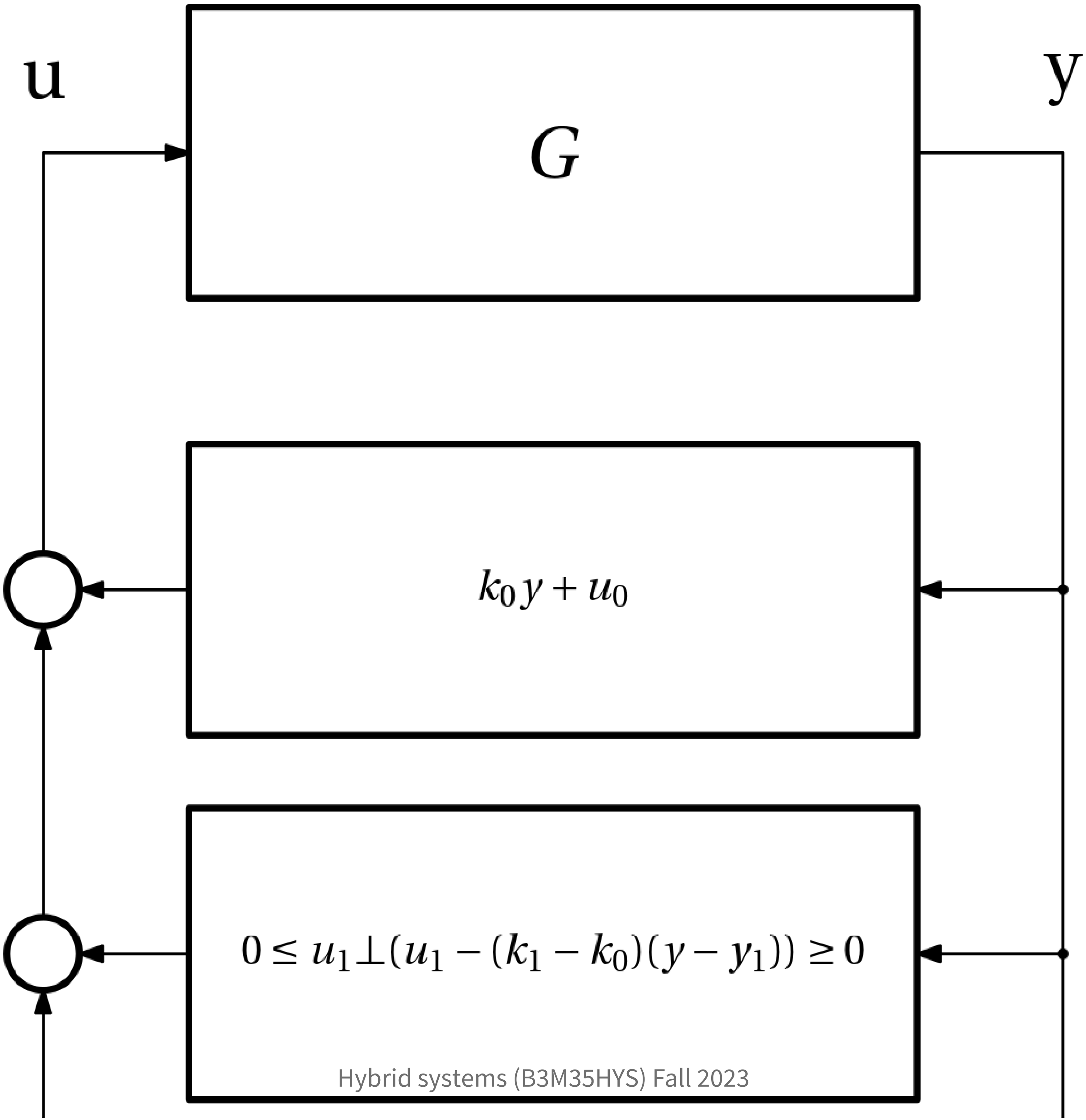
$$u(y) = k_1 \max(y - y_1, 0) = \max(k_1(y - y_1), 0)$$


u
G
y
$0 \leq u \perp (u - k_1(y - y_1)) \geq 0$

Some more segments

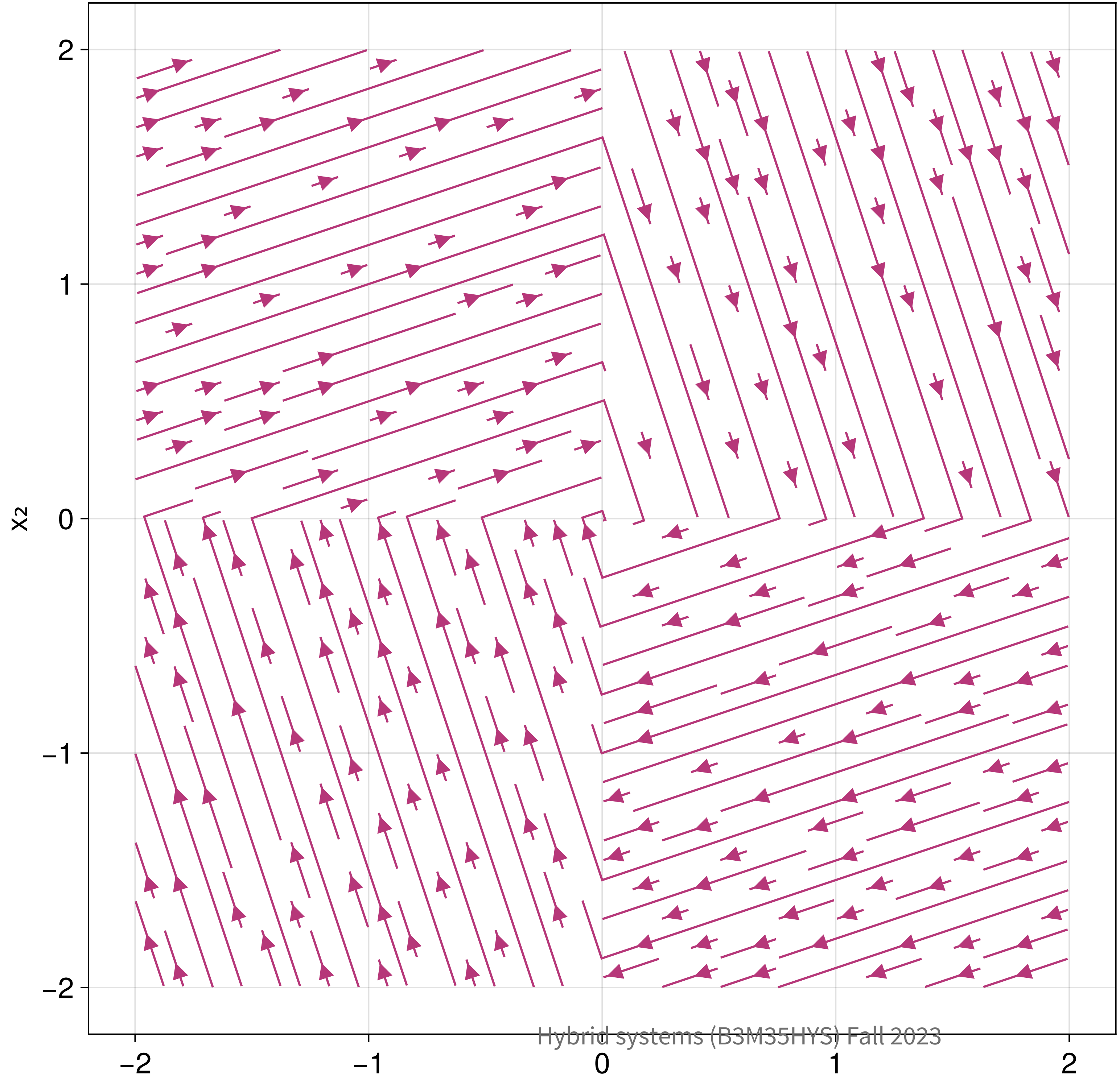


$$
\begin{aligned}
u(y) &= k_0 y + u_0 + (k_1 - k_0)\max(y - y_1, 0) \\
&\quad + (k_2 - k_1)\max(y - y_2, 0) \\
&= k_0 y + u_0 + \underbrace{\max((k_1 - k_0)(y - y_1), 0)}_{u_1} \\
&\quad + \underbrace{\max((k_2 - k_1)(y - y_2), 0)}_{u_2}
\end{aligned}
$$


u
y
G
$k_0 y + u_0$
$0 \leq u_1 \perp (u_1 - (k_1 - k_0)(y - y_1)) \geq 0$

Simulation using time-stepping

$$
\begin{aligned}
\dot{x}_1 &= -\operatorname{sign} x_1 + 2 \operatorname{sign} x_2\\
\dot{x}_2 &= -2 \operatorname{sign} x_1 - \operatorname{sign} x_2
\end{aligned}
$$


2
1
0
−1
−2
x2
−2
−1
0
1
2

Example continued: how fast does the solution approach the origin?

- Let's use the 1-norm $\|\boldsymbol{x}\|_1 = |x_1| + |x_2|$ to measure how far the trajectory is from the origin.
- How fast does the trajectory converge to the origin? That, is

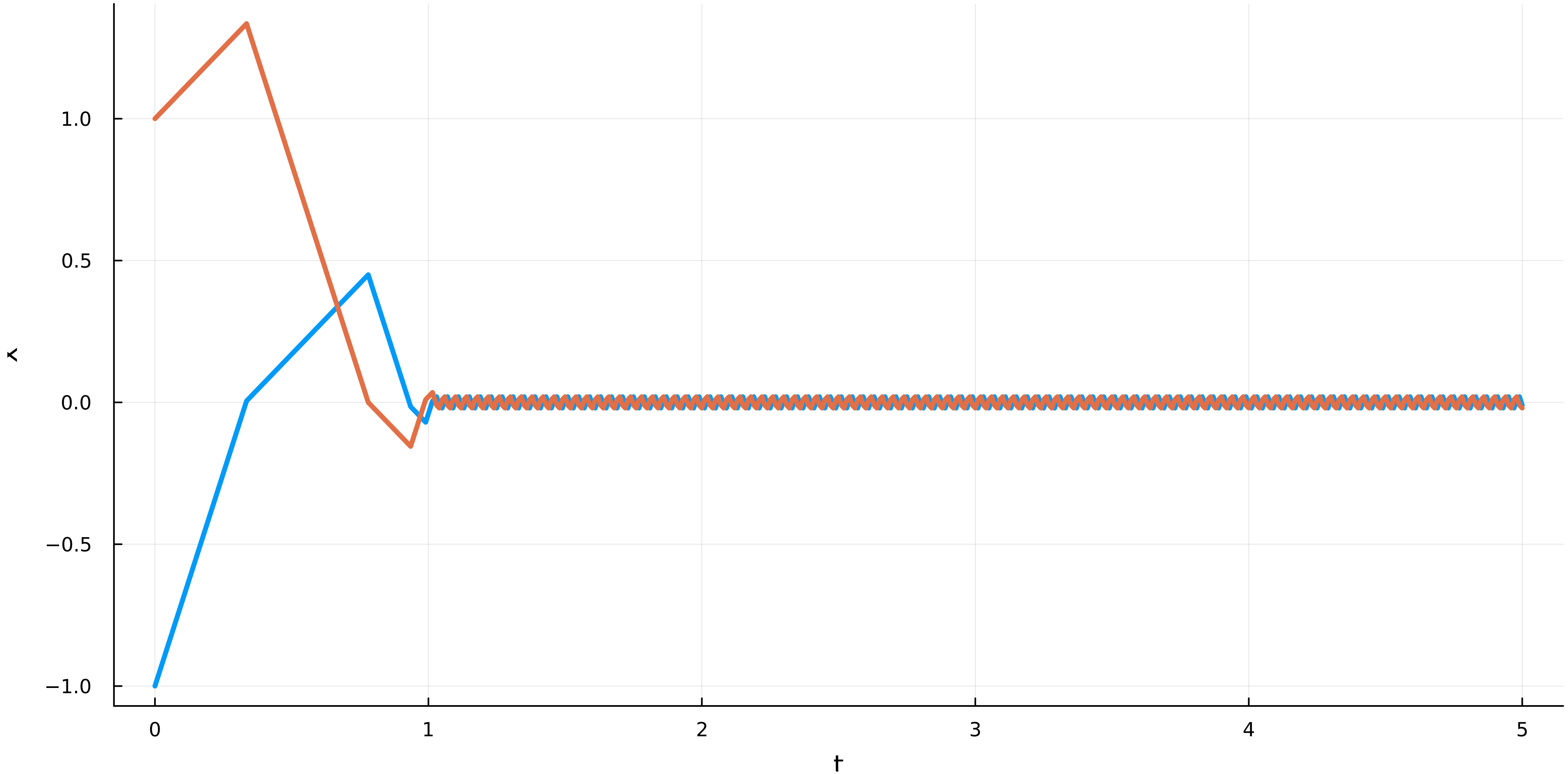
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{x}\|_1 = ?$$

- Consider each quadrant separately. Let's start in the first (upper right) quadrant, that is, $x_1 > 0$ and $x_2 > 0$, and therefore $|x_1| = x_1,\ |x_2| = x_2$, and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{x}\|_1 = \dot{x}_1 + \dot{x}_2 = 1 - 3 = -2.$$

Forward Euler with fixed step size

$$
\begin{aligned}
{\color{blue}x_{1,k+1}} &= x_{1,k} + h(-\operatorname{sign} x_{1,k} + 2\operatorname{sign} x_{2,k})\\
{\color{blue}x_{2,k+1}} &= x_{1,k} + h(-2\operatorname{sign} x_{1,k} - \operatorname{sign} x_{2,k})
\end{aligned}
$$


1.0
0.5
0.0
−0.5
−1.0
x
0
1
2
3
4
5
t

Backward Euler

$$
\begin{aligned}
{\color{blue}x_{1,k+1}} &= x_{1,k} + h(-\operatorname{sign} {\color{blue}x_{1,k+1}} + 2 \operatorname{sign} {\color{blue}x_{2,k+1}})\\
{\color{blue}x_{2,k+1}} &= x_{1,k} + h(-2 \operatorname{sign} {\color{blue}x_{1,k+1}} - \operatorname{sign} {\color{blue}x_{2,k+1}})
\end{aligned}
$$

Formulation using LCP

- Instead solving the above nasty equations, introduce new variables u_1 and u_2 as the outcomes of the sign functions:

$$
\begin{aligned}
x_{1,k+1} &= x_{1,k} + h(-u_1 + 2u_2) \\
x_{2,k+1} &= x_{1,k} + h(-2u_1 - u_2)
\end{aligned}
$$

- But now we have to enforce the **relation** between $\boldsymbol{u}$ and $\boldsymbol{x}_{k+1}$.
- Recall the standard definition of the sign function:

$$
\mathrm{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}
$$

- Change the definition to a set-valued function

$$\begin{cases} \mathrm{sign}(x) = 1 & x > 0 \\ \mathrm{sign}(x) \in [-1, 1] & x = 0 \\ \mathrm{sign}(x) = -1 & x < 0 \end{cases}$$

- Accordingly, set the relation between $\boldsymbol{u}$ and $\boldsymbol{x}$

$$\begin{cases} u_1 = 1 & x_1 > 0 \\ u_1 \in [-1, 1] & x_1 = 0 \\ u_1 = -1 & x_1 < 0 \end{cases}$$

and

$$\begin{cases} u_2 = 1 & x_2 > 0 \\ u_2 \in [-1, 1] & x_2 = 0 \\ u_2 = -1 & x_2 < 0 \end{cases}$$

- But these are *mixed complementarity contraints*!

$$\begin{bmatrix} x_{1,k+1} \\ x_{1,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + h \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$-1 \leq u_1 \leq 1 \quad \perp \quad -x_{1,k+1}$$

$$-1 \leq u_2 \leq 1 \quad \perp \quad -x_{2,k+1}$$

9 possible combinations

- Let's explore some: $x_{1,k+1} = x_{2,k+1} = 0$, while $u_1 \in [-1, 1]$ and $u_2 \in [-1, 1]$:

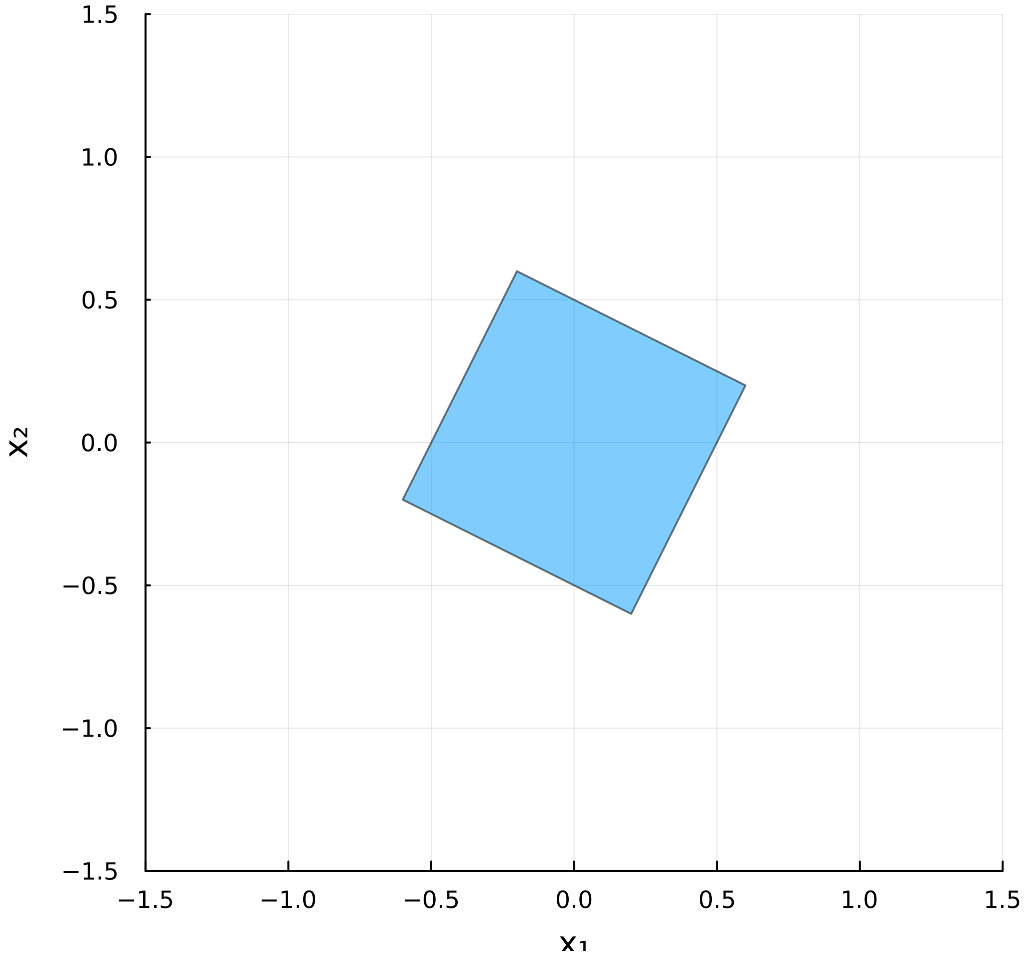
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + h \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$-1 \leq u_1 \leq 1, \quad -1 \leq u_2 \leq 1$$

- How does the set of states from which the next state is zero look like?

$$-\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = h \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$-1 \leq u_1 \leq 1, \quad -1 \leq u_2 \leq 1$$

$$\begin{bmatrix} -h \\ -h \end{bmatrix} \leq \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 0.2 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \leq \begin{bmatrix} h \\ h \end{bmatrix}$$

- For $h = 0.2$



- Indeed, if the current state is in this rotated square, then the next state will be zero.

Another

- $u_1 = 1, u_2 = 1$:

$$\begin{bmatrix} x_{1,k+1} \\ x_{1,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + h \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_{1,k+1} \geq 0$$

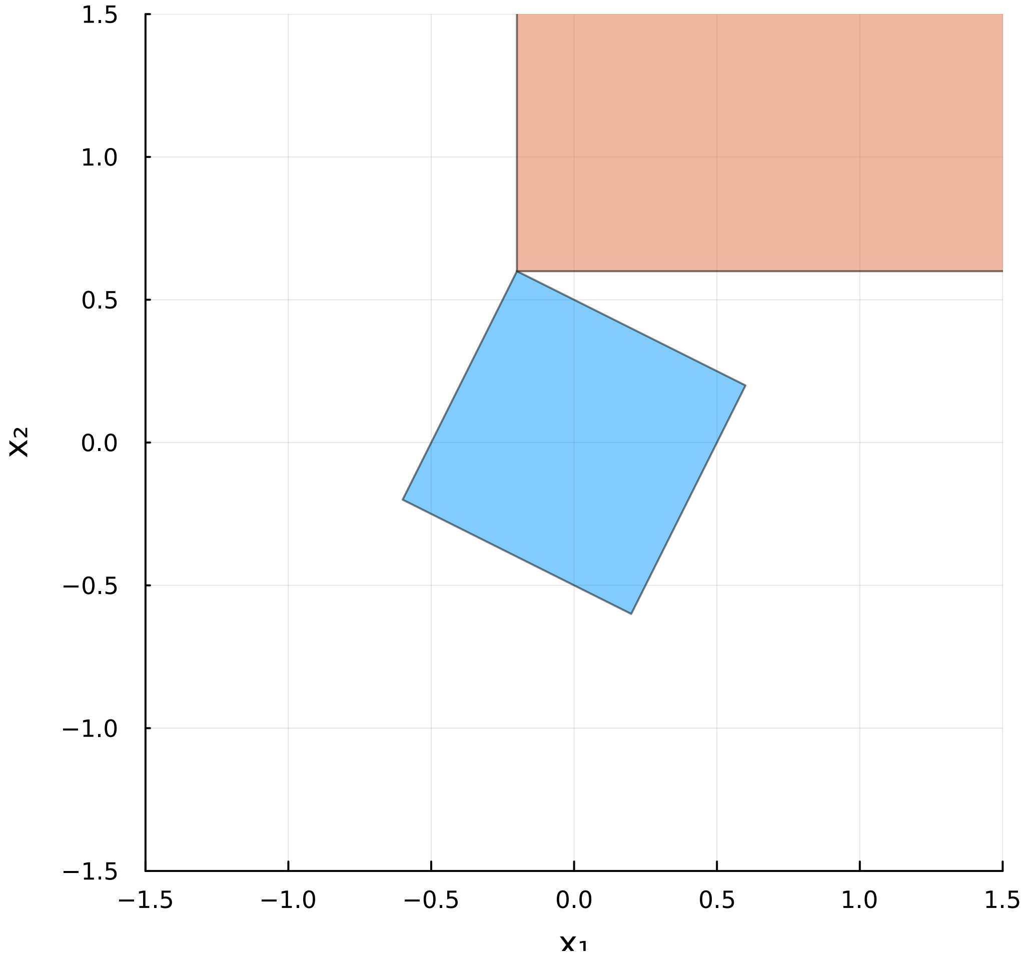
$$x_{2,k+1} \geq 0$$

- which can be reformatted to

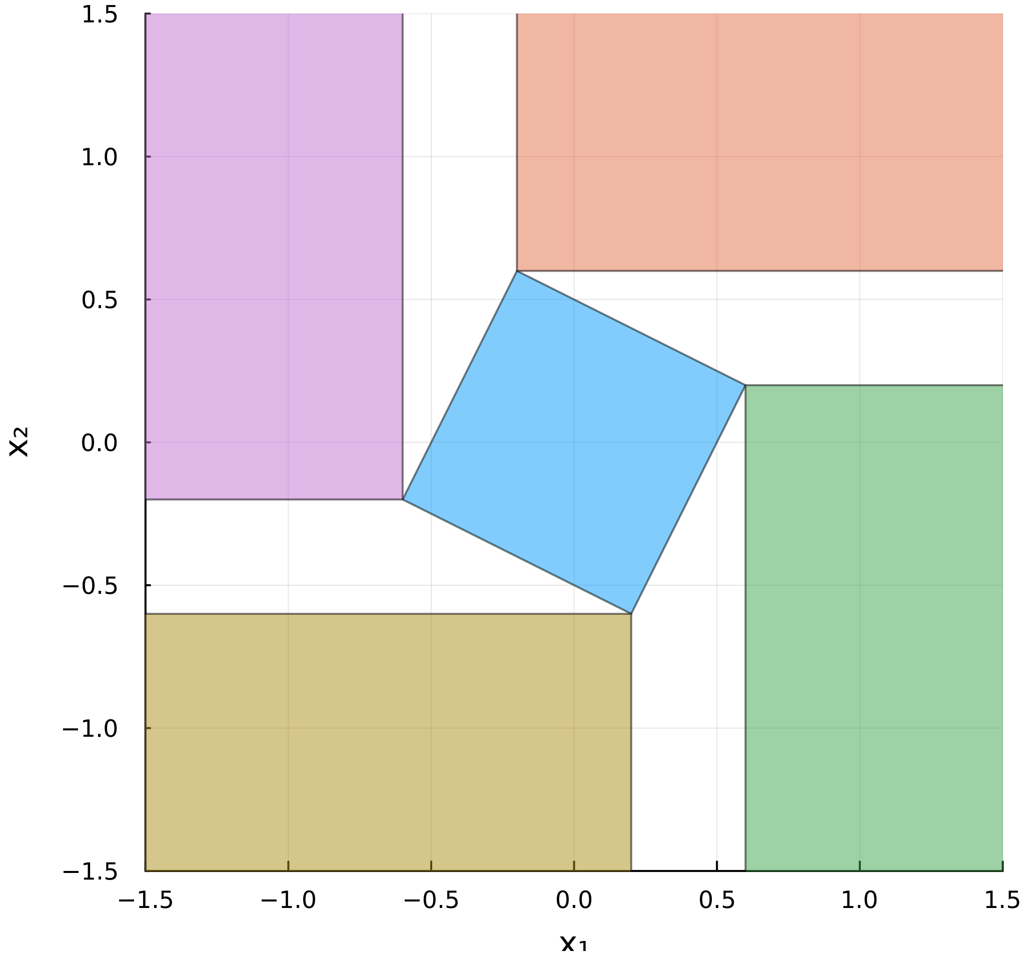
$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + h \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq \mathbf{0}$$

- and further to

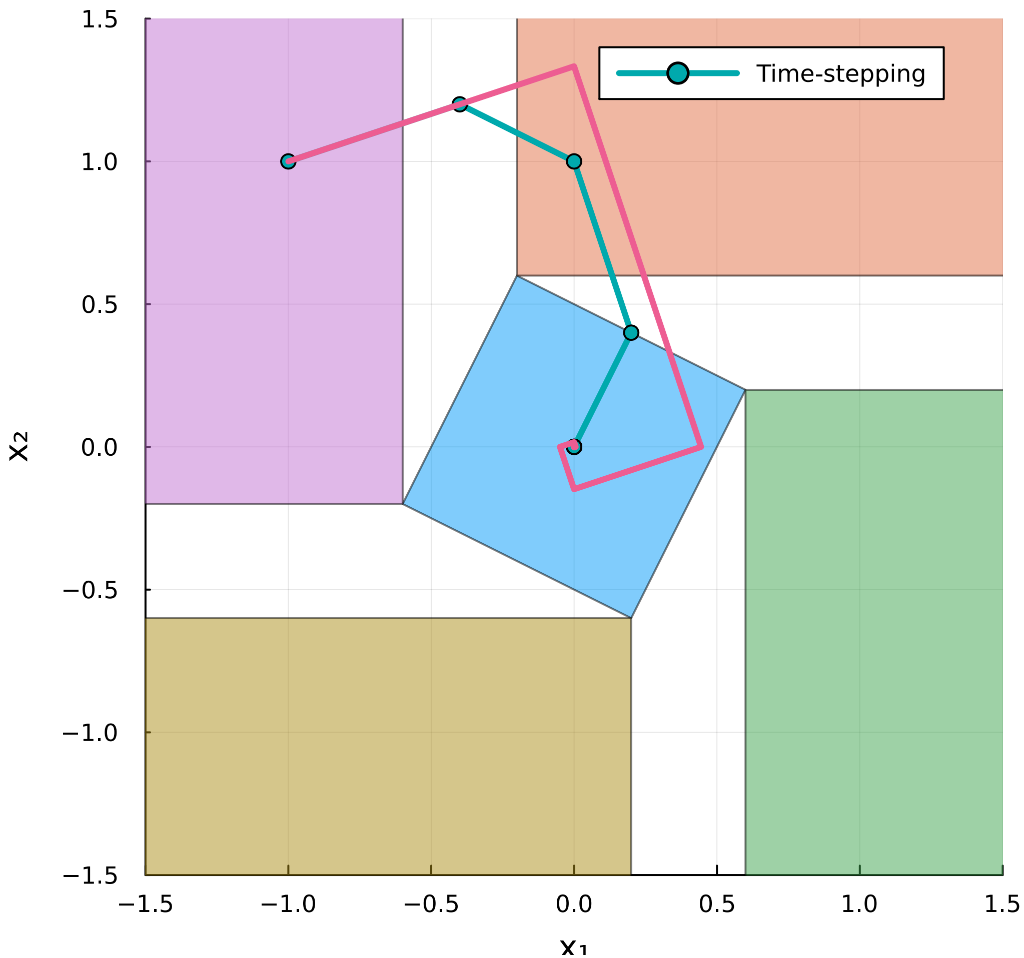
$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \geq h \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$


1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
x2
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x1

All nine regions



Solutions using a MCP solver



SW for modeling and simulation based on complementarity constraints

- SICONOS: https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/
  - C++, Python
  - physical domain independent
- PINOCCHIO: https://stack-of-tasks.github.io/pinocchio/
  - C++, Python
  - specialized for robotics